In many areas of commerce, the government intervenes in order to achieve a common good. We have discussed the financial incentives to clean up an oil spill quickly that might be used by a governmental agency interested in minimizing the environmental damage caused by oil along a coastline, and in the process we explored arguments about whether certain fine structure was overly punitive or was simply the minimal fine sufficient to achieve a timely cleanup.

In this module, we explore a similar set of questions related to ongoing pollution from an industry. Typically, factory production costs the industry money, but the sale of the goods generates revenue, and business make decisions about production to maximize revenue minus cost. They might not take the environmental effects of production required to (unless there is an advertising benefit from having a reputation as environmentally friendly), but the government can put in place various incentives to change the economics of the production decisions.

One option is to decide on a fair target for pollution levels of each business in an industry, and then fine companies that exceed their allotted pollution levels. This is bound to generate many complaints from businesses about why their level is unfair given their size, location, particular other costs or challenges, and other factors.

Another option is to implement a tax at a fixed rate per ton of pollution, and attempt to set it at a level where the overall pollution gets reduced by the desired level, but let individual companies make their own decisions about how to respond with their own production levels and investment in cleaner production methods. We now explore one such taxing mechanism, what economic decisions it causes a business to make, and its effect on pollution by that business.
Problem on taxing emissions from steel factory.

Show an elmo problem C1:0 discuss it. (underlying processes) {Q: Cow with taxing we decrease emissions?}

Variables:
- \( x \) - number of units of steel produced [unit]
- \( p = 400 - 4 \cdot x \) - market price for steel [$/unit]
- \( R = p \cdot x \) - revenue [$]
- \( C = 1000 + 10 \cdot x \) - cost to produce \( x \) units [$].
- \( E = 0.005 + 0.015 \cdot \exp (-\frac{x}{100}) \) pollutant emission per unit of steel [pollutant/unit]

\( T = E \cdot x \cdot TR \) - taxed paid if tax rate is TR for pollutants [pollutant/unit][unit][pollutant]

\( Z \) - investments in clean technology [$]

\( P = R - C - Z - T \) - profit = revenue - expenses. [$]

Do Maple worksheet

\( \uparrow \) until \( TR = 1000 \)

case of no tax on emissions.
constraint problem
no investments in clean technology.
Production \( x = 48.75 \)

Profit \( P = 8506.25 \)

Total Emissions \( E = 0.975 \)

case of emissions being taxed \( TR = 1000 \)

Maple until "comments on numerical procedure."

We see that emissions decreased

Emissions \( 1000 = 0.3393 \ldots \)

Production and profit decreased too:

\[ x = 47.86 \]

\[ P(1000) = 7966.67 \]

--- We've seen that profit dropped by 67%, while emissions dropped by 65%.

--- \( Z = 197 \), so investment in clean technology changed the results.

--- Profit dropped by 539.58

--- So about 37% of lost profit went in technology improvements to decrease emissions.

Rmk.

We used in-built exact solver to solve:

\[
\begin{align*}
\frac{\partial P}{\partial x} &= 0 \\
\frac{\partial P}{\partial z} &= 0
\end{align*}
\]

Equations were not complicated, but in more realistic situation we'll need numerical solver.

When algebraic equations are sufficiently smooth can use multivariable version of Newton's method.
Differentiable functions (Newton's method)

\[
\begin{align*}
\begin{cases}
    f_1(x_1, \ldots, x_n) = 0 \\
    \vdots \\
    f_n(x_1, \ldots, x_n) = 0
\end{cases}
\text{ or } \quad F(x) = 0
\end{align*}
\]

initial guess \((x_1(0), \ldots, x_n(0))\)

near \(x = x(0)\) linearize:

\[
F(x) \approx F(x(0)) + A \cdot (x - x(0))
\]

where

\[
A \approx \begin{pmatrix}
    \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
    \vdots & \ddots & \vdots \\
    \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}
\]

To find next iteration value:

Set \(F(x_1) = 0 \implies F(x_0) + A \cdot (x_1 - x_0) = 0\)

\[
x_1 = x_0 - A^{-1} \cdot F(x_0)
\]

in general

\[
x_n = x_{n-1} - A^{-1} \cdot F(x_{n-1})
\]
Algorithm: NEWTON'S METHOD IN TWO VARIABLES

Variables: 
- \(x(n)\) = approximate \(x\) coordinate of the root after \(n\) iterations.
- \(y(n)\) = approximate \(y\) coordinate of the root after \(n\) iterations.
- \(N\) = number of iterations

Input: \(x(0), y(0), N\)

Process: Begin
for \(n = 1\) to \(N\) do
Begin
\(q \leftarrow \delta F/\delta x(x(n-1), y(n-1))\)
\(r \leftarrow \delta F/\delta y(x(n-1), y(n-1))\)
\(s \leftarrow \delta G/\delta x(x(n-1), y(n-1))\)
\(t \leftarrow \delta G/\delta y(x(n-1), y(n-1))\)
\(u \leftarrow -F(x(n-1), y(n-1))\)
\(v \leftarrow -G(x(n-1), y(n-1))\)
\(D \leftarrow qr - rs\)
\(x(n) \leftarrow x(n-1) + (ut - vr)/D\)
\(y(n) \leftarrow y(n-1) + (qu - sv)/D\)
End
End

Output: \(x(N), y(N)\)

Figure 3.13: Pseudocode for Newton's method in two variables.

we have: \[
\begin{cases}
F(x, y) = 0 \\
G(x, y) = 0
\end{cases}
\]

\[
\begin{pmatrix}
x_n \\
y_n
\end{pmatrix} = \begin{pmatrix}
x_{n-1} \\
y_{n-1}
\end{pmatrix} - \begin{pmatrix}
\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\
\frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}
\end{pmatrix}^{-1} \begin{pmatrix}
F(x_{n-1}, y_{n-1}) \\
G(x_{n-1}, y_{n-1})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
x_{n-1} \\
y_{n-1}
\end{pmatrix} + \begin{pmatrix}
q & r
\end{pmatrix}^{-1} \begin{pmatrix}
u
\end{pmatrix}
\]

\[
= \begin{pmatrix}
x_{n-1} \\
y_{n-1}
\end{pmatrix} \left(1 + \frac{1}{qrt - rs} \begin{pmatrix}
t & -r \\
-s & q
\end{pmatrix} \begin{pmatrix}
u
\end{pmatrix}\right)
\]

\[
= \begin{pmatrix}
x_{n-1} \\
y_{n-1}
\end{pmatrix} + \frac{1}{qrt - rs} \begin{pmatrix}
utt - rv \\
-su + qv
\end{pmatrix}
\]

in the above pseudocode explicit formulas are used.
Maple starting from "comments on numerical procedure".

- With contour plot we can visually estimate the location of extremum.
- For numerical solution we use fsolve
  mix of global search & local solver
  For local solver a version of Newton's method is used.

- We explicitly coded Newton's method
  \[ \Rightarrow \text{results the same} \]

- Visual representation of first few iterations shows quick convergence to the root.

Observation.

Yes, taxing industry we can decrease emissions.
Decreasing emissions we can lessen the effect of greenhouse gases
on climate change.

We taxed company at a fixed rate per ton of pollution. Company
could decide on its own how much money to invest in cleaner
technology and what the production level should be. Under the specified
conditions we obtained optimal values. (See additional problems.)