

3. Which of the following sets are nonempty?

- (a)  $\{r \in \mathbb{Q} \mid r^2 = 2\}$
- (b)  $\{r \in \mathbb{R} \mid r^2 + 5r - 7 = 0\}$
- (c)  $\{t \in \mathbb{Z} \mid 6t^2 - t - 1 = 0\}$

In (a) we use the fact that  $\sqrt{2}$  is irrational.

- (a)  $\{r \in \mathbb{Q} \mid r^2 = 2\} = \{\pm\sqrt{2}\} \cap \mathbb{Q} = \emptyset$
- (b)  $\{r \in \mathbb{R} \mid r^2 + 5r - 7 = 0\} = \{(-5 \pm \sqrt{53})/2\} \neq \emptyset$
- (c)  $\{t \in \mathbb{Z} \mid 6t^2 - t - 1 = 0\} = \{1/2, -1/3\} \cap \mathbb{Z} = \emptyset.$

Only the set given in (b) is nonempty. ◆

12c. Find the image of the function  $f$  when  $f : \mathbb{R} \rightarrow \mathbb{R}; f(x) = -x^2 + 1$ .

A real number  $y$  is in the image of  $f$  iff  $y = -x^2 + 1$  for some  $x \in \mathbb{R}$ . Thus  $y$  is in the image of  $f$  iff  $x^2 = 1 - y$  for some  $x \in \mathbb{R}$ , that is, exactly when  $1 - y \geq 0$ . Hence, we have

$$\text{Im } f = \{y \in \mathbb{R} \mid 1 - y \geq 0\} = \{y \in \mathbb{R} \mid y \leq 1\};$$

Thus,  $\text{Im } f = (-\infty, 1]$ . ◆

25d. Prove that the given function is injective  $f : \mathbb{R} \rightarrow \mathbb{R}; f(x) = -3x + 5$ .

Suppose that for some  $a, b \in \mathbb{R}$ , we have  $f(a) = f(b)$ , that is,

$$-3a + 5 = -3b + 5.$$

Subtracting 5 from both sides yields  $-3a = -3b$ ; then dividing both sides by  $-3$  gives  $a = b$ . Hence,  $f$  is injective. ◆

29. (a) Let  $f : B \rightarrow C$  and  $g : C \rightarrow D$  be functions such that  $g \circ f$  is surjective. Prove that  $g$  is surjective.

To show that  $g$  is surjective we must show that for every  $d \in D$  there exists  $c \in C$  such that  $d = g(c)$ . So let  $d \in D$  be given. Since  $g \circ f$  is surjective, there exists  $b \in B$  such that  $d = (g \circ f)(b)$ . Now set  $c = f(b)$ , then  $c \in C$  and

$$d = (g \circ f)(b) = g(f(b)) = g(c).$$

Hence,  $g$  is surjective. ◆

(b) Give an example in part (a) in which  $f$  is not surjective.

Set  $B = C = \mathbb{R}$  and  $D = [-1, 1]$ . Define  $f : B \rightarrow C$  by  $f(x) = x^2$  and  $g : C \rightarrow D$  by  $g(x) = \sin x$ . Then  $g \circ f$  (and also  $g$ ) is surjective but  $f$  is not (since  $-1 \notin \text{Im } f$ ).

Here is a simpler example: set  $B = C = \{0, 1\}$  and  $D = \{0\}$ ; define  $f : B \rightarrow C$  by  $f(x) = 0$  for all  $x \in B$  and  $g : C \rightarrow D$  by  $g(x) = 0$  for all  $x \in C$ . Then  $g \circ f$  is surjective but not  $f$ ; for  $1 \in C$  but there is no  $x \in B$  such that  $f(x) = 1$ . ◆