Math 331: HW 11 Solutions
Tuesday, 21 Nov 2006

7.5: 2ab, 3, 15; 7.6: 5b, 15, 24;

7.5.2 List the distinct right cosets of K in G.

(a) $K = \{r_0, v\}$; $G = D_4$

By consulting the last row of the table for D_4 on page 167 to find r_i for $i = 1, 2, 3, 4$, we obtain four distinct right cosets of $K = \{r_0, v\}$:

$$K = \{r_0, v\}, \quad Kr_1 = \{r_1, t\}, \quad Kr_2 = \{r_2, h\}, \quad Kr_3 = \{r_3, d\}.$$

The collection of cosets may written succinctly as $\{Kr_i \mid i = 1, 2, 3, 4\}$.

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(b) $K = \{r_0, r_1, r_2, r_3\}$; $G = D_4$

There are two distinct cosets of $K = \{r_0, r_1, r_2, r_3\}$:

$$K = \{r_0, r_1, r_2, r_3\}, \quad Kd = \{d, h, t, v\}.$$

Note that $Kd = G - K$.

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7.5.3 Find the index $[G : H]$ when

(a) $H = \{r_0, r_2\}$; $G = D_4$

Note that $|H| = 2$ and $|G| = |D_4| = 4$. So by Lagrange's Theorem, $[G : H] = |G|/|H| = 8/2 = 4$.

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(b) $H = \langle 3 \rangle$ and $G = \mathbb{Z}_{12}$

Note that $H = \langle 3 \rangle = \{0, 3, 6, 9\}$ so $|H| = 4$. Since $|G| = |\mathbb{Z}_{12}| = 12$, we have by Lagrange's Theorem, $[G : H] = |G|/|H| = 12/4 = 3$.

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(c) $H = \langle 3 \rangle$ and $G = \mathbb{Z}_{20}$

Note that since $7 \cdot 3 \equiv 1 \bmod 20$, we have $1 \in \langle 3 \rangle = H$. Hence,

$$G = \langle 1 \rangle \subseteq \langle 3 \rangle = H.$$

Since, $H \subseteq G$, we have $H = G$ and hence $[G : H] = 1$.

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(d) H is the subgroup generated by 12 and 20 and $G = \mathbb{Z}_{40}$

The subgroup genrated by 12 and 20 contains 4 and since 12 and 20 are multiples of 4 we have

$$H = \langle 4 \rangle = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36\}.$$

Since $|H| = |\langle 4 \rangle| = 10$ and $|G| = |\mathbb{Z}_{40}| = 40$, we have by Lagrange's Theorem, $[G : H] = |G|/|H| = 40/10 = 4$.

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(e) H is the subgroup generated by $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$ and $G = S_4$

Note that $|G| = |S_4| = 4! = 24$. Let $a = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$ then $H = \langle a \rangle$. Since $|a| = 4$, we have $|H| = 4$ and so by Lagrange's Theorem, $[G : H] = |G|/|H| = 24/4 = 6$.

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7.5.15 (a) If $n > 2$, prove that U_n contains an element of order 2. [*Hint:* What are the solutions of $x^2 = 1$ in $\mathbb{Z}_n$?]

Let $n > 2$. Then $(n-1)^2 \equiv 1 \bmod n$ but $n - 1 \not\equiv 1 \bmod n$. Hence $n - 1 \in U_n$ and its order is 2.

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(b) If $n > 2$, prove that the order of the group U_n is even.

Let $n > 2$. By Corollary 7.27a, if G is a finite group, then the order of every element of G divides the order of G. Since U_n is a finite group with an element of order 2 (namely, $n - 1$), 2 divides the order of U_n. Hence, the order of the group U_n is even.

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7.6.5b Consider

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}.$$

Assuming that G is a group and that N is a subgroup of G (by part (a)), use Theorem 7.34 to show that N is normal in G.

By Theorem 7.34 it suffices to show that $gNg^{-1} \subseteq N$ for all $g \in G$, that is, we must show that for all $g \in G$ and $n \in N$, $gng^{-1} \in N$. So let $g \in G$ and $n \in N$, then $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $n = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$ for some $a, b, c, d \in \mathbb{R}$ with $ad \neq 0$. Then we compute

$$gng^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/ad \\ 0 & 1/d \end{pmatrix} = \begin{pmatrix} 1 & ac/d \\ 0 & 1 \end{pmatrix}.$$

Hence, $gng^{-1} \in N$ (since $ac/d \in \mathbb{R}$).

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7.6.15 Let $f : G \to H$ be a homomorphism of groups and let $K = \{a \in G \mid f(a) = e_H\}$. Prove that K is a normal subgroup of G.

We first show that K is a subgroup of G (this was shown in class a week or so ago but we will reprove it here). Note that K is nonempty since $f(e_G) = e_H$ so $e_G \in K$. Let $k_1, k_2 \in K$; then since f is a homomorphism we have

$$f(k_1k_2) = f(k_1)f(k_2) = e_He_H = e_H.$$

Hence, $k_1k_2 \in K$ and so K is closed under multiplication. Now let $k \in K$; then

$$f(k^{-1}) = f(k)^{-1} = e_H^{-1} = e_H.$$

Hence, $k^{-1} \in K$ and so K is closed under taking inverses. Thus, K is a subgroup. To prove that K is normal it suffices (by Theorem 7.34) to show that $gkg^{-1} \in K$ for all $g \in G$ and $k \in K$. So let $g \in G$ and $k \in K$; then

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)e_Hf(g)^{-1} = f(g)f(g)^{-1} = e_H.$$

Hence, $gkg^{-1} \in K$ and therefore K is a normal subgroup of G. ◆

7.6.24 Let H be a subgroup of order n in a group G. If H is the only subroup of order n, prove that H is normal. [*Hint:* Theorem 7.34 and Exercise 13 in Section 7.4.]

Suppose that H is the only subroup of order n. Let $a \in G$. Then by Exercise 7.4.13(a), $a^{-1}Ha$ is a subgroup of G. Since $|H| = n$, H is finite so by Exercise 7.4.13(b) we have $|a^{-1}Ha| = |H| = n$. By assumption H is the only subroup of order n. Hence, $a^{-1}Ha = H$. Therefore, H is normal by Theorem 7.34. ◆