2.1.11. Find all solutions of each congruence:

(b) $3x \equiv 1 \pmod 7$

We have $3 \cdot 5 \equiv 1 \pmod 7$. So by Theorem 2.2, if $x \equiv 5 \pmod 7$ then $3x \equiv 1 \pmod 7$. That is x is a solution if x is an element of the set

$$[5] = \{\dots, -9, -2, 5, 12, 19, \dots\}.$$

To see that all solutions are of this form we observe that if $3x \equiv 1 \pmod 7$ then

$$x \equiv 15x \equiv 5(3x) \equiv 5 \cdot 1 \equiv 5 \pmod 7.$$

Hence all solutions are of this form and the set of solutions is the congruence class [5]. Using methods of 2.3 we also may see that solutions will be unique modulo 7 since $(3, 7) = 1$. ◆

(d) $6x \equiv 10 \pmod{15}$

There are no solutions to this congruence since $6x \equiv b \pmod{15}$ only if $b = 0, 6, 12, 3, 9$. ◆

2.1.21. Prove or disprove: If $[a] = [b]$ in $\mathbb{Z}_n$, then $(a, n) = (b, n)$.

The statement is true so we will prove it. Let $a, b, n \in \mathbb{Z}$ with $n > 0$. Suppose that $[a] = [b]$. Then $a \equiv b \pmod n$ and so $a - b = nk$ for some $k \in \mathbb{Z}$ (since $n \mid a - b$). Thus, we have $a = nk + b$. So by Lemma 1.7,

$$(a, n) = (n, b) = (b, n).$$

Hence, $(a, n) = (b, n)$ as required. ◆

2.2.2. The set $\mathbb{Z}_n$ contains only n elements. To solve an equation in $\mathbb{Z}_n$ you need only substitute these n elements in the equation to see which ones are solutions. Solve these equations:

(b) $x^4 = 1$ in $\mathbb{Z}_5$.

We compute x^4 for all $x \in \mathbb{Z}_5$.

x	0	1	2	3	4
x^4	0	1	1	1	1

Hence, there are four solutions $x = 1, 2, 3, 4$. ◆

(d) $x^2 + 1 = 0$ in $\mathbb{Z}_{12}$.

We compute $x^2 + 1$ for all $x \in \mathbb{Z}_{12}$.

x	0	1	2	3	4	5	6	7	8	9	10	11
x^2+1	1	2	5	10	5	2	1	2	5	10	5	2

Hence, there are no solutions to the equation $x^2 + 1 = 0$ in $\mathbb{Z}_{12}$. ◆

2.2.6. Prove or disprove: If $ab = 0$ in $\mathbb{Z}_n$, then $a = 0$ or $b = 0$.

This assertion is false in general. Let $a = 2$, $b = 3$ and $n = 6$. Then $2 \cdot 3 = 0$ in $\mathbb{Z}_6$, but $a = 2 \neq 0$ and $b = 3 \neq 0$ in $\mathbb{Z}_6$. ◆

2.3.2. How many solutions does the equation $6x = 4$ have in

(a) $\mathbb{Z}_7$

Since $(6, 7) = 1$, the equation has one solution by Corollary 2.10 (or Theorem 2.11). ◆

(b) $\mathbb{Z}_8$

Since $(6, 8) = 2$ and $2 \mid 4$, the equation has two solutions by Theorem 2.11. ◆

(c) $\mathbb{Z}_9$

Since $(6, 9) = 3$ and $3 \nmid 4$, the equation has no solutions by Theorem 2.11. ◆

(d) $\mathbb{Z}_{10}$

Since $(6, 10) = 2$ and $2 \mid 4$, the equation has two solutions by Theorem 2.11. ◆