

7.1.4 Determine whether the set G is a group under the operation $*$.

(a) $G = \{2, 4, 6, 8\}$ in \mathbb{Z}_{10} ; $a * b = ab$.

It is routine to check that G is closed under multiplication and that multiplication is commutative. Since the ring \mathbb{Z}_{10} is associative under multiplication, associativity also holds in G . Note that in \mathbb{Z}_{10}

$$6 \cdot 2 = 2, \quad 6 \cdot 4 = 4, \quad 6 \cdot 6 = 6, \quad 6 \cdot 8 = 8;$$

hence, 6 is the identity. It remains to check that every element in G has an inverse in G . This follows from the fact that $4 \cdot 4 = 2 \cdot 8 = 6 \cdot 6 = 6$. Thus, G is a group under multiplication. \blacklozenge

(b) $G = \{2^x \mid x \in \mathbb{Q}\}$; $a * b = ab$.

Let $a, b \in G$. Then $a = 2^x$ and $b = 2^y$ for some $x, y \in \mathbb{Q}$ and so $ab = 2^x 2^y = 2^{x+y} \in G$ since $x + y \in \mathbb{Q}$. Hence, G is closed under multiplication. Associativity follows from the associativity of multiplication of real numbers. Since $1 = 2^0$ and $0 \in \mathbb{Q}$, G has an identity element. For $a = 2^x \in G$ where $x \in \mathbb{Q}$, we have $a^{-1} = 2^{-x} \in G$, since $-x \in \mathbb{Q}$. Thus, G is a group under multiplication. \blacklozenge

(c) $G = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod{5}\}$; $a * b = ab$.

G is not a group under multiplication because the existence of inverses axiom is not satisfied. Note that $6 \in G$ does not have an integer multiplicative inverse. \blacklozenge

7.1.14 Let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be the following matrices with complex entries:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that the set $Q = \{\mathbf{1}, \mathbf{i}, -\mathbf{1}, -\mathbf{i}, \mathbf{j}, \mathbf{k}, -\mathbf{j}, -\mathbf{k}\}$ is a group under matrix multiplication by writing out its multiplication table. Q is called the **quaternion group**. [Exercise 35 in Section 3.1 may be helpful.]

We note that constructing a table will show that Q is closed under matrix multiplication and taking inverses. Associativity follows from the associativity of matrix multiplication and Q contains an identity since $\mathbf{1} \in Q$. Straightforward computation (see Exercise 3.1.35) shows that

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -\mathbf{1}, & \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, & & \mathbf{ki} = \mathbf{j} = -\mathbf{ik}. \end{aligned}$$

Hence, we obtain the following multiplication table:

Q	1	i	j	k	-1	-i	-j	-k
1	1	i	j	k	-1	-i	-j	-k
i	i	-1	k	-j	-i	1	-k	j
j	j	-k	-1	i	-j	k	1	-i
k	k	j	-i	-1	-k	-j	i	1
-1	-1	-i	-j	-k	1	i	j	k
-i	-i	1	-k	j	i	-1	k	-j
-j	-j	k	1	-i	j	-k	-1	i
-k	-k	-j	i	1	k	j	-i	-1

The table shows that Q is closed under multiplication. Since $\mathbf{1}$ appears in each row and column symmetrically each element of Q has an inverse. Thus, Q is a group. \blacklozenge

7.1.21 Let $SL(2, \mathbb{R})$ be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Prove that $SL(2, \mathbb{R})$ is a group under matrix multiplication. It is called the **special linear group**.

For $a, b, c, d \in \mathbb{R}$, note that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. It is well known (from your linear algebra class) that if A, B are $n \times n$ matrices with real entries, then $\det(AB) = \det A \det B$. Hence, for $A, B \in SL(2, \mathbb{R})$ we have

$$\det(AB) = \det A \det B = 1 \cdot 1 = 1$$

and so $AB \in SL(2, \mathbb{R})$. Thus $SL(2, \mathbb{R})$ is closed under matrix multiplication. Associativity follows from the associativity of matrix multiplication. Note that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{R})$, so $SL(2, \mathbb{R})$ has an identity element. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, a simple computation shows that $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL(2, \mathbb{R})$. Hence, $SL(2, \mathbb{R})$ is a group under matrix multiplication. \blacklozenge

7.2.9b List the order of each element of the group U_{20} .

Note that $U_{20} = \{1, 3, 7, 9, 11, 13, 17, 19\}$. Easy computations show that the order of each element is as given in the following table:

x	1	3	7	9	11	13	17	19
$ x $	1	4	4	2	2	4	4	2

7.2.18 (a) Show that $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3 in $GL(2, \mathbb{R})$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4.

Note that

$$a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad a^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad a^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and

$$b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, a has order 3 and b has order 4 in $GL(2, \mathbb{R})$.

(b) Show that ab has infinite order.

We compute

$$ab = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix};$$

it is straightforward to show by induction that for all $n \in \mathbb{N}$ we have

$$(ab)^n = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}.$$

Hence, $(ab)^n \neq e$ for all $n > 0$, where e is the identity matrix, and thus ab has infinite order.

7.2.22 If $(ab)^2 = a^2b^2$ for all $a, b \in G$, prove that G is abelian.

Suppose that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Let $a, b \in G$, then by multiplying the given equation on the left by a^{-1} and on the right by b^{-1} we have

$$ab = a^{-1}a(ab)bb^{-1} = a^{-1}(a^2b^2)b^{-1} = a^{-1}((ab)^2)b^{-1} = a^{-1}(ab)(ab)b^{-1} = a^{-1}a(ba)bb^{-1} = ba.$$

Hence, $ab = ba$ and so G is abelian.