Math 331 - Groups, Rings and Fields
Review sheet

Final Exam Friday, 15 Dec, 7:30 am
Monday, 11 Dec 2006

The Final Exam will cover all the material in the course thru secton 8.1. There will be a review session in class on Tuesday and another review session in AB 638, 2:15-3:45, Thursday, 14 December. Please go over past review sheets and homework problems as well as the text. You will permitted one page of notes (but please do not include worked out problems). The problems here only deal with material from sections 7.7, 7.8, 8.1

(1) Show that $\mathbb{Z}_{24}/N \cong \mathbb{Z}_6$, where N is the suibgroup $\{0, 6, 12, 18\}$.
(2) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_8$ and let N be the cyclic subgroup generated by $(1, 2)$. Show that $G/N \cong \mathbb{Z}_8$.
(3) Prove that every element of the group $\mathbb{Q}/\mathbb{Z}$ has finite order. Show that for every positve integer n, there is an injective homomorphism $f : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$.
(4) Let $f : G \to H$ be a homomorphism of groups and let $K = \{a \in G \mid f(a) = e_H\}$. Prove that there is an isomorphism $\varphi : G/K \to \operatorname{Im} f$ such that $\varphi(Ka) = f(a)$ for all $a \in G$.
(5) Let $\mathbb{C}^*$ be the multiplicative group of nonzero complex numbers. Prove that $H = \{z \in \mathbb{C}^* \mid |z| = 1\}$ is a normal subgroup of $\mathbb{C}^*$. Show that H has elements of order n for all $n > 0$; does H have elements of infinite order? Prove that $C^*/H \cong R^{**}$ (the multiplicative group of positive real numbers) and that $H \cong \mathbb{R}/\mathbb{Z}$.
(6) Let G be the set of all matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$. Prove that G is a group and that the subset N consisting of such matrices with $a = 0$ constitutes a normal subgroup. Prove that $N \cong \mathbb{Z} \times \mathbb{Z}$ and that $G/N \cong \mathbb{Z}$.
(7) Let G be a group and let $f : G \to \mathbb{Z}$ be a surjective homomorphism. Prove that G is abelian, if $K_f \subseteq Z(G)$, where K_f is the kernel of f.
(8) Suppose that G_1 and G_2 are groups and let $N_i \subseteq G_i$ be a normal subgroup for $i = 1, 2$. Show that $N_1 \times N_2$ is a normal subgroup of $G_1 \times G_2$ and that

$$(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2.$$

(9) Let $f_1 : G \to H_1$ and $f_2 : G \to H_2$ be homomorphisms of groups such that f_1 is surjective and $K_1 \subseteq K_2$ where K_i is the kernel of f_i for $i = 1, 2$. Prove that there is a homomorphism $\varphi : H_1 \to H_2$ so that $f_2 = \varphi \circ f_1$. Is φ unique?
(10) Write $\mathbb{Z}_{21}$ as the direct product of two of its subgroups.
(11) Consider

$$G = \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \,\middle|\, a, b, d \in \mathbb{R} \text{ and } a, d > 0 \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}.$$

Prove that G is a group and that N is a normal subgroup. Prove that $G/N \simeq \mathbb{R}^{**} \times \mathbb{R}^{**}$. Let

$$K = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \text{ and } a > 0 \right\}.$$

Show that K is normal in G and that $G/K \cong \mathbb{R}^{**}$. Prove that $G \cong K \times \mathbb{R}^{**}$.
[*Hint:* Find a normal subgroup H of G such that $H \cong \mathbb{R}^{**}$, $G = HK$ and $H \cap K = \{I\}$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.]
(12) Show that $Z(D_4) \cong \mathbb{Z}_2$ and that $D_4/Z(D_4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(13) Suppose that G, G_1, and G_2 are abelian groups and that $f_i : G_i \to G$ is a homomorphism for $i = 1, 2$. Show that the map $f : G_1 \times G_2 \to G$ given by $f(g_1, g_2) = f_1(g_1)f_2(g_2)$, where $g_i \in G_i$ for $i = 1, 2$, is a homomorphism. Prove that f is injective, if f_i is injective for $i = 1, 2$ and $\operatorname{Im} f_2 \cap \operatorname{Im} f_2 = \{e\}$. Under what circumstances will f be surjective? Under what circumstances will f be an isomorphism?
(14) Let G be the set of matrices of the form $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ where $\theta \in \mathbb{R}$. Prove that G is a group which is isomorphic to $\mathbb{R}/\mathbb{Z}$.