

The test will cover sections Appenices A, B and Chapters 1, 2 and 3. There will be a review session in class on Tuesday and another review session in AB 638, 4:15-5:30, Wednesday, 11 Oct. Most of the following problems require carefully worded proofs. Even problems of a more “computational” nature require written justification for the methods employed. Please review past homework problems as well as the text. You will permitted one page of notes (but please do not include worked out problems).

- (1) Suppose that X is a nonempty subset of \mathbb{Z} . State the negation of the following assertion (in the simplest possible form). Give an example of an X for which the statement is true and another example for which the statement is false.

For every $x \in X$, there is a $y \in X$, such that $x < y$.

- (2) Let n be an odd integer. Show that $n^2 = 8k + 1$ for some integer k .
(3) Is every integral domain a field?
(4) Prove that the cube of an integer must be of the form $9k$, $9k + 1$ or $9k + 8$ for some integer k .
(5) Give examples of functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$, if possible, such that $g \circ f = 1_X$ but neither f nor g is a bijection.
(6) Find all solutions to each of the following congruences:

$$4x + 6 \equiv 3 \pmod{10} \quad 4x + 7 \equiv 3 \pmod{11} \quad 4x + 7 \equiv 3 \pmod{12}$$

- (7) Let $f : [1, \infty) \rightarrow [-4, \infty)$ be given by $f(x) = x^2 - 2x - 3$. Show that f is a bijection and find its inverse.
(8) Find all solutions to the congruence $x^2 \equiv x \pmod{6}$.
(9) Use the Euclidean algorithm to find the greatest common divisor $d = \gcd(360, 378)$ and integers s, t such that $d = 360s + 378t$. Express the fraction $378/360$ in lowest terms. Find the largest negative integer in the set $\{360u + 378v \mid u, v \in \mathbb{Z}\}$.
(10) Use the Fundamental Theorem of Arithmetic to find the greatest common divisor and least multiple of the integers 360 and 378.
(11) Find the prime factorization of 237600.
(12) Let a, b, c be integers such that $a \neq 0$. Prove that if $a \mid b$ and $a \mid c$, then for all $s, t \in \mathbb{Z}$, $a \mid (bs + ct)$.
(13) Show that the congruence $x^3 \equiv x + 1 \pmod{6}$ has no solution.
(14) Find all solutions to the equation $x^2 + x = 0$ in \mathbb{Z}_6 .
(15) Let a, b, c be positive integers. Show that $c = (a, b)$ if and only if there exist $u, v \in \mathbb{Z}$ such that $c = au + bv$ and c is a common divisor of both a and b .
(16) Let a, b, c be positive integers and suppose that $d = (a, b)$ is the greatest common divisor of a, b . Show that $d \mid c$ if and only if there exist integers s, t such that $c = sa + tb$.
(17) Let a, b, c be positive integers. Show that $\gcd(ab, ac) = a \gcd(b, c)$.
(18) Let a, b, c be integers such that a and b are not both 0 and $(a, b) = 1$. Show that if $a \mid c$ and $b \mid c$, then $ab \mid c$.
(19) Prove that the product of three consecutive integers is divisible by 6.
(20) Show that the inverse of an isomorphism $f : R \rightarrow S$ is also an isomorphism.
(21) Show that the intersection of two subrings of a ring is a subring but that the union of two subrings of a ring is not necessarily a subring. Under what conditions will this be true?
(22) Let n be a positive integer and let p be a positive prime. Show that there exist unique integers k, m such that $k \geq 0$, $(p, n) = 1$ and $n = p^k m$.
(23) Let R be an integral domain and let $a, b, c \in R$. Show that $(a - b)(a - c) = 0$ if and only if $a = b$ or $a = c$.
(24) Let $f : R \rightarrow S$ be a surjective homomorphism of rings. Show that S is commutative if R is (but not necessarily conversely).
(25) Let R and S be rings and let $f : R \rightarrow S$ be a homomorphism. Show that if T is a subring of S , then $Q = \{r \in R \mid f(r) \in T\}$ is a subring of R .
(26) Show that $S = \{0, 2, 4, 6, 8\}$ is a subring of \mathbb{Z}_{10} . Use a tables to show that S is isomorphic to \mathbb{Z}_5 . Use this isomorphism to define a homomorphism $f : \mathbb{Z}_5 \rightarrow \mathbb{Z}_{10}$.
(27) Let R and S be rings with identity and let $f : R \rightarrow S$ be a homomorphism. Prove that $f(1_R) = 1_S$ or find a counterexample. If $u \in R$ is a unit, must $f(u)$ also be a unit? What if we require that $f(1_R) = 1_S$?
(28) Show that the subset S of $M(\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is a subring of $M(\mathbb{Z})$. Find a surjective homomorphism $f : S \rightarrow \mathbb{Z}$.
(29) For positive integers m, n , let $f : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ be “defined” by $f([a]_n) = [a]_m$ for $a \in \mathbb{Z}$. Show that this is well defined if and only if $m \mid n$. And that in this case f is a surjective homomorphism.
(30) Let a, b, c, n be integers with $n > 0$ and $a \not\equiv 0 \pmod{n}$. Does $ab \equiv ac \pmod{n}$ imply $b \equiv c \pmod{n}$? If so prove it and if not give a counterexample.
(31) Let $\mathbb{Z}[\sqrt[3]{2}]$ denote the set $\{a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \mid a, b, c \in \mathbb{Z}\}$. Show that $\mathbb{Z}[\sqrt[3]{2}]$ is a subring of \mathbb{R} .
(32) Find, if possible, a nonzero homomorphism $f : \mathbb{Z} \rightarrow \mathbb{Z}_6 \times \mathbb{Z}_4$. Is f surjective?
(33) Find, if possible, a nonzero homomorphism $f : \mathbb{Z}_5 \rightarrow \mathbb{Z}_6$. If you think it is impossible, please give a reason.
(34) Give an example, if possible, of a noncommutative ring with zero divisors.
(35) Show that $\mathbb{Q}[i]$ is a subfield of \mathbb{C} . (Note $\mathbb{Q}[i] = \{a + bi \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$).