1.2. Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on $x$.

(b) $B = \{0, 1, 2, 3\}$

Note that $B$ consists of all the nonnegative elements of $S$. Hence, we have $B = \{x \in S : x \geq 0\}$. □

(d) $D = \{-2, 2, 3\}$

An element $x \in S$ is an element in $D$ exactly when $|x| \geq 2$. Hence, we have $D = \{x \in S : |x| \geq 2\}$. □

1.4. Write each of the following sets by listing its elements within braces.

(b) $B = \{n \in \mathbb{Z} : n^2 < 5\}$

Note that an integer $n \in \mathbb{Z}$ is an element of $B$ exactly when $|n| < \sqrt{5}$, that is, $|n| \leq 2$. Therefore, $B = \{-2, -1, 0, 1, 2\}$. □

(d) $D = \{x \in \mathbb{R} : x^2 - x = 0\}$

The equation $x^2 - x = 0$ has two real solutions: $x = 0, 1$. So $D = \{0, 1\}$. □

1.6(c) The set $E = \{2x : x \in \mathbb{Z}\}$ can be described by listing its elements, namely $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. List the elements of the following set in a similar manner.

$$C = \{3q + 1 : q \in \mathbb{Z}\}$$

Setting $q = -2, -1, 0, 1, 2$ we obtain $3q + 1 = -5, -2, 1, 4, 7$. This establishes a clear pattern and we have

$$C = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}.$$

□

1.10 Which of the following sets are equal?

$$A = \{n \in \mathbb{Z} : |n| < 2\} \qquad D = \{n \in \mathbb{Z} : n^2 \leq 1\}$$
$$B = \{n \in \mathbb{Z} : n^3 = n\} \qquad E = \{-1, 0, 1\}$$
$$C = \{n \in \mathbb{Z} : n^2 \leq n\}$$

All of the above sets with exception of $C$ are equal. Let $n$ be an integer. Then $n \in A$ exactly when $|n| \leq 1$, that is, when $n = -1, 0, 1$. Similarly, $n^3 = n$ precisely when $n = -1, 0, 1$; and $n^2 \leq 1$ exactly when $|n| \leq 1$, i.e., when $n = -1, 0, 1$. But $n^2 \leq n$ fails to hold for $n = -1$ and only holds for $n = 0, 1$. Hence, $A = B = D = E$ but $C = \{0, 1\} \neq E$. □

1.14 Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

We have

$$\begin{aligned} \mathcal{P}(\{1\}) &= \{\varnothing, \{1\}\}, \\ \mathcal{P}(\mathcal{P}(\{1\})) &= \mathcal{P}(\{\varnothing, \{1\}\}) \\ &= \{\varnothing, \{\varnothing\}\, \{\{1\}\}, \{\varnothing, \{1\}\}\}. \end{aligned}$$

Since $\mathcal{P}(\mathcal{P}(\{1\}))$ has four elements, we have $|\mathcal{P}(\mathcal{P}(\{1\}))| = 4$. □

1.18(b) Give examples of three sets $A$, $B$ and $C$ such that

$$B \in A, B \subset C \text{ and } A \cap C \neq \varnothing.$$

There are many possible solutions. Perhaps the simplest is given by $B = \varnothing$ and $A = C = \{\varnothing\}$. Then $B$ is certainly an element of $A$ as well as a proper subset of $C$. Moreover, $A \cap C = \{\varnothing\} \neq \varnothing$.

Alternatively, take $A = \{\{1\}\}$, $B = \{1\}$, $C = \{1, \{1\}\}$. Then $B \subset C$ and since $A = \{B\}$, $B \in A$. Moreover, $A \subseteq C$, so $A \cap C = A \neq \varnothing$. □

1.22 Give an example of a universal set $U$, two sets $A$ and $B$, and an accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = \left|\overline{A \cup B}\right| = 2$.

Let

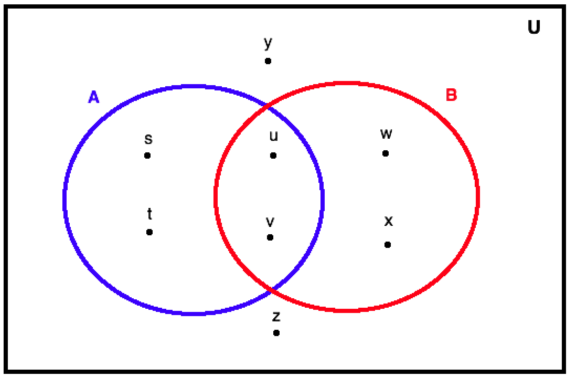
$$U = \{s, t, u, v, w, x, y, z\},$$
$$A = \{s, t, u, v\},$$
$$B = \{u, v, w, x\}.$$

Then

$$A \cap B = \{u, v\},$$
$$A - B = \{s, t\},$$
$$B - A = \{w, x\},$$
$$\overline{A \cup B} = \{y, z\}.$$

Each of these sets has two elements. □

Venn diagram




1.28 For a real number $r$, define $S_r$ to be the interval $[r-1, r+2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

Note that $S_1 = [0, 3]$, $S_3 = [2, 5]$ and $S_4 = [3, 6]$ and so

$$\bigcup_{\alpha \in A} S_\alpha = S_1 \cup S_3 \cup S_4 = [0, 3] \cup [2, 5] \cup [3, 6] = [0, 6],$$
$$\bigcap_{\alpha \in A} S_\alpha = S_1 \cap S_3 \cap S_4 = [0, 3] \cap [2, 5] \cap [3, 6] = \{3\}.$$

□