Math 373 - HW 12 solutions 10.16, 10.18, 11.4, 11.6;
Tuesday, 30 Nov 2010

10.16 Prove that the set of complex numbers is uncountable.

Recall that $\mathbb{R}$, the set of real numbers, is a subset of $\mathbb{C}$, the set of complex numbers. By Corollary 10.10, $\mathbb{R}$ is uncountable. Then since $\mathbb{R} \subseteq \mathbb{C}$, it follows by Theorem 10.9 that $\mathbb{C}$ is also uncountable. $\square$

10.18 (a) Prove that the function $f : (0,1) \to (0,2)$, mapping the open interval $(0,1)$ into the open interval $(0,2)$ and defined by $f(x) = 2x$, is bijective.

Let $x, y \in (0,1)$ and suppose that $f(x) = f(y)$. Then $2x = 2y$; dividing by both sides by 2, yields $x = y$. Hence, f is injective. Now let $y \in (0,2)$ be given. Setting $x = y/2$ (note that $x \in (0,1)$) we have

$$f(x) = 2x = 2(y/2) = y.$$

Hence, f is surjective. Since f is both injective and surjective, f is bijective. We could also have proven this by finding an inverse for f. $\square$

(b) Explain why $(0,1)$ and $(0,2)$ have the same cardinality.

Two sets have the same cardinality if and only if there is bijection between. Since we have shown that $f : (0,1) \to (0,2)$ as defined above is a bijection, $(0,1)$ and $(0,2)$ have the same cardinality. $\square$

(c) Let $a, b \in \mathbb{R}$, where $a < b$. Prove that $(0,1)$ and (a,b) have the same cardinality.

We define a function $g : (0,1) \to (a,b)$ by $g(x) = (b-a)x + a$. Note that for every $x \in (0,1)$ and $y \in (a,b)$, we have

$$y = (b-a)x + a \quad \text{if and only if} \quad x = \frac{y-a}{b-a}.$$

It follows that the function $h : (a,b) \to (0,1)$ defined by $h(y) = \frac{y-a}{b-a}$ is the inverse of g, i.e., $h = g^{-1}$. So by Theorem 9.11, g is bijective. Since $g : (0,1) \to (a,b)$ defines a one-to-one correspondence, $(0,1)$ and (a,b) have the same cardinality. $\square$

11.4 Prove that $3 \mid (n^3 - n)$ for every integer n.

In our proof below, we use the fact that for every $a, b \in \mathbb{R}$,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Let $n \in \mathbb{Z}$ be given. Then there exist (unique) integers q, r such that $n = 3q + r$ and $0 \le r < 3$. Thus there are three cases to consider: $r = 0, 1, 2$.

i. Suppose that $r = 0$. Then $n = 3q$ and

$$n^3 - n = (3q)^3 - 3q = 3q(9q^2 - 1).$$

Since $n^3 - n = 3m$ where $m = q(9q^2 - 1) \in \mathbb{Z}$, we have $3 \mid (n^3 - n)$.

ii. Suppose that $r = 1$. Then $n = 3q + 1$ and

$$n^3 - n = (3q+1)^3 - (3q+1) = 27q^3 + 27q^2 + 9q + 1 - (3q+1) = 3q(9q^2 + 9q + 2).$$

Since $n^3 - n = 3m$ where $m = q(9q^2 + 9q + 2) \in \mathbb{Z}$, we have $3 \mid (n^3 - n)$.

iii. Suppose that $r = 0$. Then $n = 3q$ and

$$n^3 - n = (3q+2)^3 - (3q+2) = 27q^3 + 54q^2 + 36q + 8 - (3q+2) = 3(9q^3 + 18q^2 + 11q + 2).$$

Since $n^3 - n = 3m$ where $m = 9q^3 + 18q^2 + 11q + 2 \in \mathbb{Z}$, we have $3 \mid (n^3 - n)$.

Hence, in all cases, $n^3 \equiv n \pmod 3$ and so the desired result follows. $\square$

11.6 Find all primes that are 1 less than a perfect cube.

We show that there is only one prime of this form, namely, 7. Observe that $7 = 2^3 - 1$ (and, of course, 7 is a prime). Let $n \in \mathbb{Z}$ and suppose that $p = n^3 - 1$ is prime. Then since $p \ge 2$, we have $n^3 - 1 > 0$. Hence, $n > 1$. We claim that $n < 3$. Assume to the contrary that $n \ge 3$. Then

$$p = n^3 - 1 = (n-1)(n^2 + n + 1).$$

Since $n \ge 3$, we have $n - 1 > 1$ and $n^2 + n + 1 > 1$. Hence, p is composite. This contradicts the fact that p is prime. Hence, the claim follows and $1 < n < 3$. So $n = 2$ and $p = 7$. $\square$