Math 373 – Review sheet **Final Exam**: Friday, 10 Dec, 7:30 am.

Sun, 5 Dec 2010 / A Kumjian Review: Tues, 7 Dec

The Final Exam will be comprehensive with some emphasis on the material not covered in Tests I and II, that is, sections 6.2, 6.3, 8.1 – 8.6, 9.1 – 9.6, 10.1 – 10.3, 11.1 – 11.5. You may bring a formula sheet to the final exam. It may include definitions, results, propositions etc., but please do not include sample proofs or other worked out problems. There will be a supplementary office hour (or review) Thursday afternoon 4-5:15 pm.

To receive full credit, solutions must be correct, complete and neatly written; give reasons for your answers. Review the previous tests, review sheets, quizzes and all assigned homework problems (and their solutions). The most recent quiz problems are listed below.

1. Use mathematical induction to prove that $7n + 5 < 2^n$ for every integer $n \geq 6$.
2. Let R be the relation defined on $\mathbb{Z}$ by $x\ R\ y$ if $|x - y| \leq 3$. Which of the properties, reflexive, symmetric and transitive, does the relation R possess? Justify your answers. Is R an equivalence relation?
3. Let R be the relation defined on $\mathbb{Z}$ by $a\ R\ b$ if $a^2 \equiv b^2 \pmod 4$. Given that R is an equivalence relation, determine the distinct equivalence classes. [Hint: For all $a, b \in \mathbb{Z}$, if $a \equiv b \pmod 4$ then $a^2 \equiv b^2 \pmod 4$].
4. Construct a multiplication table for $\mathbb{Z}_5$ with entries of the form $[r]$ for $r = 0, 1, 2, 3, 4$. Use the table to prove that for every $[a] \in \mathbb{Z}_5$, if $[a] \neq [0]$, then there is $[b] \in \mathbb{Z}_5$ such that $[a] \cdot [b] = [1]$. [Hint: there are four cases.]
5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = 2n^2 + 1$.
   (a) Determine whether f is one-to-one.
   (b) Determine whether f is onto.
6. Prove that the function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{1\}$ defined by $f(x) = 1 + \frac{1}{x}$ is bijective.
7. Show that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{0\}$ defined by $f(x) = \frac{2}{x-3}$, for $x \in \mathbb{R} - \{3\}$, has an inverse and determine $f^{-1}(x)$ for $x \in \mathbb{R} - \{0\}$.
8. Let $S \subseteq \mathbb{N} \times \mathbb{N}$ be defined by $S = \{(m, n) : m \geq n\}$. Prove that S is denumerable.

Here are some other sample problems:

9. Prove that $2^{n+1} < n!$ or every integer $n \geq 5$.
10. Prove that for every integer $n \geq 13$, there exist $j, k \in \mathbb{N}$ such that $n = 3j + 4k$.
11. If possible, find a relation on a nonempty set A such that R is symmetric and transitive but not reflexive.
12. Let R be the relation on $\mathbb{Z}$ defined by $m\ R\ n$ if $3m \equiv 3n \pmod{12}$. Prove that R is an equivalence relation, determine the distinct equivalence classes.
13. Let $S = \mathbb{N} \times \mathbb{N}$. Let R be the relation on S defined by $(a, b)\ R\ (c, d)$ if $ad = bc$. Show that R is an equivalence relation and describe the equivalence classes $[(1, 2)]$ and $[(2, 3)]$. Can you determine the distinct equivalence classes?
14. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = 3n + 2$. Is f one-to-one? Is f onto? Would your answers be any different if the domain and codomain were both $\mathbb{R}$ instead?
15. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(n) = x^2 - 4x + 7$. Is f one-to-one? Is f onto? If f is bijective, find an inverse.
16. Let $f : [2, \infty) \to [3, \infty)$ be the function defined by $f(n) = x^2 - 4x + 7$. Is f one-to-one? Is f onto? If f is bijective, find an inverse.
17. Let $f : A \to B$ and $g : B \to A$. Prove
    (a) If $g \circ f$ is surjective, then g is surjective.
    (b) If $g \circ f$ is injective, then f is injective.
    (c) If $g \circ f$ and $f \circ g$ are bijective, then so are f and g.
    (d) Suppose $g \circ f = i_A$ and $f \circ g = i_B$. What can be said about the relationship between f and g?
18. Let $A \subseteq B$. Prove: if B is countable then A is countable. Prove that the converse is false.
19. Let A and B be denumerable sets. Prove that $A \cup B$ is denumerable.
20. Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b]$ is uncountable.
21. Let $n \in \mathbb{Z}$. Use the division algorithm to prove that if $n \equiv 5 \pmod 6$, then n is not a perfect square.
22. Use the Euclidean algorithm to find $d = \gcd(884, 1037)$ and $x, y \in \mathbb{Z}$ such that $d = 884x + 1037y$.
23. Let $n \in \mathbb{Z}$. Prove that $n \equiv 0 \pmod{210}$ if and only if $n \equiv 0 \pmod{14}$ and $n \equiv 0 \pmod{15}$.
24. Let $n \in \mathbb{Z}$. Prove that there exist $x, y \in \mathbb{Z}$ such that $n = 55\ell + 78m$.
25. Let $n \in \mathbb{Z}$. Prove that if $55n \equiv 0 \pmod{78}$, then $n \equiv 0 \pmod{78}$.
26. Let $n \in \mathbb{Z}$ and p be a prime. Prove that if $n \not\equiv 0 \pmod p$, then there is $m \in \mathbb{Z}$ such that $mn \equiv 1 \pmod p$.
27. Let p be a prime. Prove that for every $n \in \mathbb{N}$, $p \mid n^2$ if and only if $p \mid n$. Prove that $\sqrt{p}$ is irrational.

In addition, please review the exercises:
6.13, 6.19, 6.45, 8.11, 8.15, 8.19, 8.49, 8.59, 9.15, 9.17, 9.21, 9.33, 9.49, 10.1, 10.7, 10.9, 10.15, 11.13, 11.15, 11.27, 11.29, 11.31, 11.37, 11.43.