Math 373 - Quiz 11 solutions

Thursday, 2 Dec 2010

1. Show that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{0\}$ defined by

$$f(x) = \frac{2}{x-3}, \qquad \text{for } x \in \mathbb{R} - \{3\},$$

has an inverse and determine $f^{-1}(x)$ for $x \in \mathbb{R} - \{0\}$.

To prove that $f$ has an inverse, one could show that $f$ is bijective in the standard ways; but since we must also find a formula for the inverse it would be easier to define a function $g : \mathbb{R} - \{3\} \to \mathbb{R} - \{0\}$ and then show that it is the inverse.

Let $g : \mathbb{R} - \{3\} \to \mathbb{R} - \{0\}$ be defined by

$$g(x) = \frac{2}{x} + 3, \qquad \text{for } x \in \mathbb{R} - \{0\}.$$

Let $x \in A = \mathbb{R} - \{3\}$. Then we have

$$(g \circ f)(x) = g(f(x)) = \frac{2}{f(x)} + 3 = \frac{2}{\frac{2}{x-3}} + 3 = \frac{2(x-3)}{2} + 3 = x;$$

hence, $g \circ f = i_A$. Let $x \in B = \mathbb{R} - \{0\}$. Then we have

$$(f \circ g)(x) = f(g(x)) = \frac{2}{g(x) - 3} = \frac{2}{\left(\frac{2}{x} + 3\right) - 3} = \frac{2}{\frac{2}{x}} = x;$$

hence, $f \circ g = i_B$. It follows that $f$ has an inverse and that

$$f^{-1}(x) = \frac{2}{x} + 3,$$

for $x \in \mathbb{R} - \{0\}$.

Another approach is to proceed as follows: Let $x \in \mathbb{R} - \{0\}$ and let $y \in \mathbb{R} - \{3\}$. Then

$$\begin{aligned} y &= \frac{2}{x-3} && \text{if and only if} \\ x - 3 &= \frac{2}{y} && \text{if and only if} \\ x &= \frac{2}{y} + 3. \end{aligned}$$

Thus $y = f(x)$ if and only if $x = g(y)$ where $g : \mathbb{R} - \{3\} \to \mathbb{R} - \{0\}$ is the function defined by

$$g(x) = \frac{2}{x} + 3, \qquad \text{for } x \in \mathbb{R} - \{0\}.$$

Hence, $f$ has an inverse and $f^{-1} = g$. □

2. Let $S \subseteq \mathbb{N} \times \mathbb{N}$ be defined by $S = \{(m, n) : m \geq n\}$. Prove that $S$ is denumerable.

Since the cartesian product of two denumerable sets is denumerable, $\mathbb{N} \times \mathbb{N}$ is denumerable. Since an infinite subset of a denumerable set is itself denumerable, it suffices to show that $S$ is infinite. Let $T = \{(n, n) : n \in \mathbb{N}\}$, then $T$ is an infinite subset of $S$, so $S$ must also be infinite. Hence, $S$ is denumerable.

Alternatively, one may define a bijection $f : \mathbb{N} \to S$ which is suggested by listing the elements of $S$ as follows:

$$S = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), \ldots\}.$$

It follows that $\mathbb{N}$ and $S$ have the same cardinality, that is, $S$ is denumerable. □