Math 373 - Test II Solutions
Thursday, 28 Oct 2010

Each problem is worth ten points. Please write clearly and neatly. These problems all require proofs, which must be written in complete English sentences. For full credit please *show all steps and give reasons.* Your cheat sheet may not include sample proofs or worked out problems.
Please attempt no more than five of the following six problems. Clearly indicate the problem you omit.

1. Let $n \in \mathbb{Z}$. Prove that if $5n+3$ is even, then n is odd.

   Proof. We prove this statement by contrapositive. Suppose that n is even. Then $n = 2k$ for some integer k. Hence
   $$5n + 3 = 5(2k) + 3 = 10k + 3 = 2(5k + 1) + 1.$$
   Since, $m = 5k + 1 \in \mathbb{Z}$, $5n + 3 = 2m + 1$ is odd. The desired result now follows. □

2. Disprove the following statement: There is a real number x such that
   $$x^2 + 3 \leq 3x.$$

   Proof. To disprove this statement, we must prove its negation: For every real number x,
   $$x^2 + 3 > 3x.$$
   Let $x \in \mathbb{R}$. Then
   $$x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} > \left(x - \frac{3}{2}\right)^2 \geq 0.$$
   Hence, $x^2 - 3x + 3 > 0$ and thus $x^2 + 3 > 3x$. This disproves the stated result. □

3. Let $a, b, n \in \mathbb{Z}$ and suppose that $n \geq 2$. Use the definition of congruence to prove that $a + b \equiv 0 \pmod{n}$ implies $a^2 \equiv b^2 \pmod{n}$.

   Proof. Suppose that $a + b \equiv 0 \pmod{n}$. Then $n \mid (a + b)$ and so $a + b = nk$ for some integer k. Observe that
   $$a^2 - b^2 = (a + b)(a - b) = nk(a - b).$$
   Since $m = k(a - b)$ is an integer, $a^2 - b^2 = nm$ is divisible by n, that is, $n \mid (a^2 - b^2)$. Therefore, by the definition of congruence, $a^2 \equiv b^2 \pmod{n}$. □

4. Prove that $A \subseteq B$ where
   $$A = \{x \in \mathbb{R} : x - 2 \geq 0\} \quad \text{and} \quad B = \{x \in \mathbb{R} : x^2 - 1 > 0\}.$$

   Proof. Let $x \in \mathbb{R}$ and suppose that $x \in A$. Then $x - 2 \geq 0$ and so $x \geq 2$. Thus
   $$x^2 \geq 4 > 1.$$
   Since $x^2 > 1$, we have $x^2 - 1 > 0$ and therefore $x \in B$. Hence, $A \subseteq B$. □

5. Prove that $\sqrt{7}$ is irrational. [Hint: You may use the fact: $\forall n \in \mathbb{Z}$, $7 \mid n$ if and only if $7 \mid n^2$.]

   Proof. Suppose to the contrary that $\sqrt{7}$ is rational. Then $\sqrt{7} = a/b$, where $a, b \in \mathbb{Z}$ and $b \neq 0$; we may further assume that a and b have no common factors (that is a/b is a fraction expressed in lowest terms). Since $7 = a^2/b^2$, we have $a^2 = 7b^2$; moreover, $7 \mid a^2$, since $b^2 \in \mathbb{Z}$. By the hint $7 \mid a$ and since a and b have no common factors, $7 \nmid b$. So $a = 7q$ for some $q \in \mathbb{Z}$; hence,
   $$7b^2 = a^2 = (7q)^2 = 49q^2.$$
   It follows that $b^2 = 7q^2$ and since $q^2 \in \mathbb{Z}$, we have $7 \mid b^2$. Again, by the hint, $7 \mid b$. This results in a contradiction, since $7 \nmid b$. Therefore, $\sqrt{7}$ is irrational. □

6. Use mathematical induction to prove that for every positive integer n,
   $$3 + 5 + \cdots + (2n + 1) = n(n + 2).$$

   Proof. Denote this equation by $P(n)$. We use induction to prove: $\forall n \in \mathbb{N}, P(n)$.
   Base step: For $n = 1$, we have $3 = 1(2 + 1)$. Hence, $P(1)$ is true.
   Inductive step: Now suppose that that $P(k)$ holds for some $k \in \mathbb{N}$. Then
   $$3 + 7 + \cdots + (2k + 1) = k(k + 2).$$

We must now verify that $P(k+1)$ is true: by $P(k)$ we have

$$\begin{aligned}3+7+\cdots+(2k+1)+(2(k+1)+1) &= k(k+2)+(2(k+1)+1)\\ &= k^2+2k+2k+3\\ &= k^2+4k+3\\ &= (k+1)(k+3)\\ &= (k+1)((k+1)+2).\end{aligned}$$

Hence, $P(k+1)$ is true. Therefore, by the principle of mathematical induction

$$3+5+\cdots+(2n+1) = n(n+2)$$

for all $n \in \mathbb{N}$. □