Cohomology of higher rank graphs and twisted $k$-graph $C^*$-algebras, an interim report.

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We introduce the homology of a higher rank graph and discuss the corresponding cohomology.

Our definition of the homology of a $k$-graph $\Lambda$ is modeled on Massey’s cubical singular homology (see [Mas91, §VII.2]).

It is equivalent to the homology of a cubical set as defined by Grandis (see [Gr05]).

We define the twisted $C^*$-algebra $C^*_\varphi(\Lambda)$ where $\varphi$ is a $\mathbb{T}$-valued 2-cocycle.

All noncommutative tori may be realized as examples of this construction.

This is an interim report on joint work with David Pask and Aidan Sims of the University of Wollongong.
**Definition (see [KP00])**

Let $\Lambda$ be a countable small category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. Then $(\Lambda, d)$ is a $k$-graph if it satisfies the factorization property:

For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that

$$d(\lambda) = m + n$$

there exist unique $\mu, \nu \in \Lambda$ satisfying:

- $d(\mu) = m$ and $d(\nu) = n$,
- $\lambda = \mu \nu$.

Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \text{Obj}(\Lambda)$, the set of vertices. An element $\lambda \in \Lambda^{e_i}$ is called an edge.
Remarks and Examples.

Let $\Lambda$ be a $k$-graph.

- If $k = 0$, then $d$ is trivial and $\Lambda$ is just a set.
- If $k = 1$, then $\Lambda$ is the path category of a directed graph.
- If $k \geq 2$, think of $\Lambda$ as generated by $k$ graphs of different colors that share the same set of vertices $\Lambda^0$.

Commuting squares form an essential piece of structure for $k \geq 2$.

Let $C_m$ denote the directed cycle with $m$ vertices viewed as a 1-graph.

Example of a 2-graph: Only the morphisms of minimal degree, $\Lambda^{e_1}$ and $\Lambda^{e_2}$, are shown.

![Diagram](attachment:diagram.png)

Note that $\Lambda \cong C_2 \times C_1$. 
More examples

The $k$-graph $T_k := \mathbb{N}^k$ is regarded as the $k$-graph analog of a torus.

Here is a simple $k$-graph with an infinite number of vertices:

$$\Delta_k := \{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k \mid m \leq n\}$$

with structure maps

$$s(m, n) = n$$
$$r(m, n) = m$$
$$d(m, n) = n - m$$
$$(\ell, n) = (\ell, m)(m, n).$$

This may be regarded as the $k$-graph analog of Euclidean space.
Cubes and Faces.

Let $\Lambda$ be a $k$-graph. For $0 \leq n \leq k$, an element $\lambda \in \Lambda$ with

$$d(\lambda) = e_{i_1} + \cdots + e_{i_n} \quad \text{where} \quad i_1 < \cdots < i_n$$

is called an $n$-cube. Let $Q_n(\Lambda)$ denote the set of $n$-cubes.

Note that 0-cubes are vertices and 1-cubes are edges.

For $n < 0$ or $n > k$ set $Q_n(\Lambda) = \emptyset$.

Let $\lambda \in Q_n(\Lambda)$. We define the faces $F^0_j(\lambda), F^1_j(\lambda) \in Q_{n-1}(\Lambda)$, where $1 \leq j \leq n$, to be the unique elements such that

$$\lambda = F^0_j(\lambda)\lambda_0 = \lambda_1 F^1_j(\lambda)$$

where $d(\lambda_\ell) = e_{i_j}$ for $\ell = 0, 1$.

Fact: If $i < j$, then $F^\ell_i \circ F^m_j = F^m_{j-1} \circ F^\ell_i$. 
Homology complex.

For $1 \leq n \leq k$ define $\partial_n : \mathbb{Z}Q_n(\Lambda) \rightarrow \mathbb{Z}Q_{n-1}(\Lambda)$ such that for $\lambda \in Q_n(\Lambda)$

$$\partial_n(\lambda) = \sum_{j=1}^{n} \sum_{\ell=0}^{1} (-1)^{j+\ell} F_{j}^{\ell}(\lambda).$$

It is straightforward to show that $\partial_{n-1} \circ \partial_n = 0$.

Hence, $(\mathbb{Z}Q_*(\Lambda), \partial_*)$ is a complex and we define the homology of $\Lambda$ by

$$H_n(\Lambda) = \ker \partial_n / \text{Im} \partial_{n+1}.$$ 

The assignment $\Lambda \mapsto H_*(\Lambda)$ is a covariant functor.

Example: Recall that $C_m$ is a cycle with $m$ vertices. One may check that

$$H_n(C_m) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$
The Künne'sh Theorem.

Using basic homological algebra one may prove:

**Theorem (Künne'sh Formula)**

Let $\Lambda_i$ be a $k_i$-graph for $i = 1, 2$. For $n \geq 0$ there is an exact sequence:

$$0 \to \sum_{m_1 + m_2 = n} H_{m_1}(\Lambda_1) \otimes H_{m_2}(\Lambda_2) \xrightarrow{\alpha} H_n(\Lambda_1 \times \Lambda_2) \xrightarrow{\beta} \sum_{m_1 + m_2 = n-1} \text{Tor}(H_{m_1}(\Lambda_1), H_{m_2}(\Lambda_2)) \to 0.$$

Let $\Lambda$ be the 2-graph example above and recall that $\Lambda \cong C_2 \times C_1$.

By the Künne'sh Theorem we have

$$H_0(\Lambda) \cong \mathbb{Z}, \quad H_1(\Lambda) \cong \mathbb{Z}^2, \quad H_2(\Lambda) \cong \mathbb{Z}.$$
Acyclic $k$-graphs and free actions.

A $k$-graph $\Lambda$ is said to be *acyclic* if $H_0(\Lambda) \cong \mathbb{Z}$ and $H_n(\Lambda) = 0$ for $n > 0$. It is easy to show that $\Delta_k$ is acyclic.

**Theorem**

*Let $\Lambda$ be an acyclic $k$-graph and suppose that there is a free action of the group $G$ on $\Lambda$. Then for each $n \geq 0$ there is an isomorphism:*

$$H_n(\Lambda/G) \cong H_n(G).$$

*Example.* Take $\Lambda = \Delta_k$ and let $G = \mathbb{Z}^k$ act on $\Delta_k$ by translation.

We have $\Delta_k/\mathbb{Z}^k \cong T_k$ and so

$$H_n(T_k) \cong H_n(\mathbb{Z}^k) \cong \mathbb{Z}^{(k)}_n.$$ 

Of course, this also follows by the Künneth Theorem.
Cohomology.

Let $\Lambda$ be a $k$-graph and let $A$ be an abelian group. For $n \in \mathbb{N}$ set

$$C^n(\Lambda, A) = \text{Hom}(\mathbb{Z}Q_n(\Lambda), A)$$

and define

$$\delta^n : C^n(\Lambda, A) \to C^{n+1}(\Lambda, A) \quad \text{by} \quad \delta^n(\varphi) = \varphi \circ \partial_{n+1}.$$

It is straightforward to show that $(C^*(\Lambda, A), \delta^*)$ is a complex.

We define the cohomology of $\Lambda$ by

$$H^n(\Lambda, A) := Z^n(\Lambda, A)/B^n(\Lambda, A),$$

where $Z^n(\Lambda, A) := \ker \delta^n$ and $B^n(\Lambda, A) := \text{Im} \delta^{n-1}$.

Note $\Lambda \mapsto H^*(\Lambda, A)$ is a contravariant functor (it is covariant in $A$).
The UCT and a long exact sequence.

**Theorem (Universal Coefficient Theorem)**

Let \( \Lambda \) be a \( k \)-graph and let \( A \) be an abelian group. Then for \( n \geq 0 \), there is a short exact sequence

\[
0 \to \text{Ext}(H_{n-1}(\Lambda), A) \to H^n(\Lambda, A) \to \text{Hom}(H_n(\Lambda), A) \to 0.
\]

By a standard argument, a short exact sequence of coefficient groups

\[
0 \to A \to B \to C \to 0
\]

gives rise to a long exact sequence

\[
0 \to H^0(\Lambda, A) \to H^0(\Lambda, B) \to H^0(\Lambda, C) \to H^1(\Lambda, A) \to \cdots
\]
\[
\cdots \to H^{n-1}(\Lambda, C) \to H^n(\Lambda, A) \to H^n(\Lambda, B) \to H^n(\Lambda, C) \to \cdots
\]
The $C^*$-algebra $C^*_\varphi(\Lambda)$.

Suppose that $\Lambda$ satisfies $(\ast)$: For all $v \in \Lambda^0$, $n \in \mathbb{N}^k$, $v\Lambda^n$ is finite and nonempty where $v\Lambda^n := r^{-1}(v) \cap \Lambda^n$.

**Definition**

Let $\varphi \in Z^2(\Lambda, \mathbb{T})$. Define $C^*_\varphi(\Lambda)$ to be the universal $C^*$-algebra generated by a family of operators $\{t_\lambda : \lambda \in \Lambda^{e_i}, 1 \leq i \leq k\}$ and a family of orthogonal projections $\{p_v : v \in \Lambda^0\}$ satisfying:

1. For $\lambda \in \Lambda^{e_i}$, $t_\lambda^* t_\lambda = p_{s(\lambda)}$.
2. Suppose $\mu \nu = \nu' \mu'$ where $d(\mu) = d(\mu') = e_i$, $d(\nu) = d(\nu') = e_j$ and $i < j$. Then $t_{\nu'} t_{\mu'} = \varphi(\mu \nu) t_{\mu} t_{\nu}$.
3. For $v \in \Lambda^0$ and $i = 1, \ldots, k$,

$$p_v = \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^*.$$

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Cohomology of $k$-graphs
Main Results.

Fact: The isomorphism class of $C^*_\varphi(\Lambda)$ only depends on $[\varphi] \in H^2(\Lambda, \mathbb{T})$.

There is a gauge action $\gamma$ of $\mathbb{T}^k$ on $C^*_\varphi(\Lambda)$: For all $z \in \mathbb{T}^k$

\[
\begin{align*}
\gamma_z(p_v) &= p_v & \text{for all } v \in \Lambda^0, \\
\gamma_z(t_\lambda) &= z_i t_\lambda & \text{for all } \lambda \in \Lambda^{e_i}, i = 1, \ldots, k. 
\end{align*}
\]

Moreover, the fixed point algebra $C^*(\Lambda, \varphi)^\gamma$ is AF (cf. [KP00]).

Theorem (Gauge Invariant Uniqueness Theorem)

Let $\pi : C^*(\Lambda, \varphi) \to B$ be an equivariant $*$-homomorphism. Then $\pi$ is injective iff $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.

Theorem

There is a $\mathbb{T}$-valued groupoid 2-cocycle $\sigma_\varphi$ on $\mathcal{G}_\Lambda$ such that

\[
C^*_\varphi(\Lambda) \cong C^*(\mathcal{G}_\Lambda, \sigma_\varphi).
\]
Rotation algebras

Recall that $T_k = \mathbb{N}^k$.

There is precisely one 2-cube in $T_2$, namely $(1, 1)$.

Fix $\theta \in [0, 1)$. Let $\phi \in \mathbb{Z}^2(\Lambda, \mathbb{T})$ be given by $\phi(1, 1) = e^{2\pi i \theta}$.

Then $C^*_\phi(T_2)$ is the universal $C^*$-algebra generated by unitaries $S_{e_1}$ and $S_{e_2}$ satisfying

$$S_{e_2}S_{e_1} = e^{2\pi i \theta} S_{e_1}S_{e_2}.$$ 

That is, $C^*_\phi(T_2)$ is the rotation algebra $A_\theta$.

When $\theta = 0$, $C^*_\phi(T_2) \cong C(\mathbb{T}^2)$.

When $\theta$ is irrational, $C^*_\phi(T_2)$ is the well-known irrational rotation algebra.

More generally, every noncommutative torus arises as a twisted $k$-graph $C^*$-algebra $C^*_\phi(T_k)$. 
Heegaard quantum 3-spheres

The quantum 3-sphere $S_{pq\theta}^3$ where $p, q, \theta \in [0, 1)$ is defined in [BHMS]. The authors prove that $S_{pq\theta}^3 \cong S_{00\theta}^3$.

Note $S_{00\theta}^3$ is the universal $C^*$-algebra generated by $S$ and $T$ satisfying

$$
(1 - SS^*)(1 - TT^*) = 0, \quad ST = e^{2\pi i \theta} TS, \\
S^*S = T^*T = 1, \quad ST^* = e^{-2\pi i \theta} T^*S.
$$

It was known that $S_{000}^3$ is isomorphic to $C^*(\Lambda)$ where $\Lambda$ is the 2-graph given as follows:

But what about $S_{00\theta}^3$?
Quantum spheres are twisted 2-graph $C^*$-algebras

The degree map gives a homomorphism $f : \Lambda \to T_2$ and the induced map

$$f^* : H^2(T_2, \mathbb{T}) \to H^2(\Lambda, \mathbb{T}).$$

is an isomorphism.

There are three 2-cubes $\alpha = ah = hb$, $\beta = cg = fc$ and $\tau = af = fa$.

Fix $\theta \in [0, 1)$. The 2-cocycle on $T_2$ determined by $(1, 1) \mapsto e^{-2\pi i \theta}$ pulls back to a 2-cocycle $\phi$ on $\Lambda$ satisfying

$$\phi(\alpha) = \phi(\beta) = \phi(\tau) = e^{-2\pi i \theta}.$$ 

Let $\{t_\lambda : \lambda \in \Lambda^{ei}, 1 \leq i \leq k\}$ and $\{p_v : v \in \Lambda^0\}$ be the generators of $C^*_\phi(\Lambda)$.

By the universal property there is a unique map $\Psi : S^3_{000} \to C^*_\phi(\Lambda)$ such that $\Psi(S) = t_a + t_b + t_c$ and $\Psi(T) = t_f + t_g + t_h$.

Moreover, $\Psi$ is an isomorphism.
Categorical cocycle cohomology.

The categorical cocycle cohomology, $H^*_cc(\Lambda, A)$, is just the usual cocycle cohomology for groupoids (see [Ren80]) extended to small categories. We have proven that for $n = 0, 1, 2$

$$H^n(\Lambda, A) \cong H^n_{cc}(\Lambda, A).$$

A map $c : \Lambda \ast \Lambda \to A$ is a categorical 2-cocycle if for any composable triple $(\lambda_1, \lambda_2, \lambda_3)$ we have

$$c(\lambda_1, \lambda_2) + c(\lambda_1 \lambda_2, \lambda_3) = c(\lambda_1, \lambda_2 \lambda_3) + c(\lambda_2, \lambda_3)$$

and $c$ is a categorical 2-coboundary if there is $b : \Lambda \to A$ such that

$$c(\lambda_1, \lambda_2) = b(\lambda_1) - b(\lambda_1 \lambda_2) + b(\lambda_2).$$

$H^2_{cc}(\Lambda, A)$ is the quotient group (2-cocycles modulo 2-coboundaries).
The $C^*$-algebra $C^*(\Lambda, c)$.

Suppose $\Lambda$ satisfies $(\ast)$ and let $c$ be a $\mathbb{T}$-valued categorical 2-cocycle.

**Definition (see [KPS])**

Let $C^*(\Lambda, c)$ be the universal $C^*$-algebra generated by the set $\{t_\lambda : \lambda \in \Lambda\}$ satisfying:

1. $\{t_v : v \in \Lambda^0\}$ is a family of orthogonal projections.
2. For $\lambda \in \Lambda$, $t_{s(\lambda)} = t_\lambda^* t_\lambda$.
3. If $s(\lambda) = r(\mu)$, then $t_\lambda t_\mu = c(\lambda, \mu) t_{\lambda \mu}$.
4. For $v \in \Lambda^0$, $n \in \mathbb{N}^k$

\[
t_v = \sum_{\lambda \in v \Lambda^n} t_\lambda t_\lambda^*.
\]

If $[\varphi]$ is mapped to $[c]$ in the identification $H^2(\Lambda, A) \cong H^2_{cc}(\Lambda, A)$, then

\[
C^*_\varphi(\Lambda) \cong C^*(\Lambda, c).
\]
Topological realizations.

One may construct the topological realization $X_{\Lambda}$ of a $k$-graph $\Lambda$ (see [KKQS]) by analogy with the geometric realization of a simplicial set. Let $I = [0, 1]$. For $i = 1, \ldots, n$ and $\ell = 0, 1$ define $\varepsilon_i^\ell : I^{n-1} \to I^n$ by

$$\varepsilon_i^\ell (x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \ell, x_i, \ldots, x_{n-1}).$$

Then the topological realization is the quotient of

$$\bigsqcup_{n=0}^{k} Q_n(\Lambda) \times I^n$$

by the equivalence relation generated by $(\lambda, \varepsilon_i^\ell (x)) \sim (F_i^\ell (\lambda), x)$ where $\lambda \in Q_n(\Lambda)$ and $x \in I^{n-1}$.

We prove that there is a natural isomorphism $H_n(\Lambda) \cong H_n(X_{\Lambda})$. 
References.


Thanks!

Any questions?