Abstract. Let $A$ be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. We show that $\mathcal{O}_E$, the Cuntz-Pimsner algebra associated to the Hilbert $A$-bimodule $E = \mathcal{H} \otimes \mathbb{C}A$, is simple and purely infinite. If $A$ is nuclear and belongs to the bootstrap class to which the UCT applies, then the same applies to $\mathcal{O}_E$. Hence by the Kirchberg-Phillips Theorem the isomorphism class of $\mathcal{O}_E$ only depends on the $K$-theory of $A$ and the class of the unit.

In his seminal paper [Pm], Pimsner constructed a C*-algebra $\mathcal{O}_E$ from a Hilbert bimodule over a C*-algebra $A$ as a quotient of a concrete C*-algebra $\mathcal{T}_E$, an analogue of the Toeplitz algebra, acting on the Fock space associated to $E$. There has recently been much interest in these Cuntz-Pimsner algebras (or Cuntz-Krieger-Pimsner algebras), which generalize both crossed products by $\mathbb{Z}$ and Cuntz-Krieger algebras, as well as the associated Toeplitz algebras. The structure of these C*-algebras is not yet fully understood, although considerable progress has been made. For example, Pimsner found a six-term exact sequence for the $K$-theory of $\mathcal{O}_E$ which generalizes the Pimsner-Voiculescu exact sequence (see [Pm, Theorem 4.8]); conditions for simplicity were found in [Sc2, MS, KPW1, DPW] and for pure infiniteness in [Z].

The purpose of the present note is to analyze the structure of Cuntz-Pimsner algebras associated to a certain class of Hilbert bimodules. Let $A$ be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful nondegenerate representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Then $E = \mathcal{H} \otimes \mathbb{C}A$ is a Hilbert bimodule over $A$ in a natural way. We show that $\mathcal{O}_E$ is separable, simple and purely infinite. If $A$ is nuclear and in the bootstrap class, then the same holds for $\mathcal{O}_E$ and thus by the Kirchberg-Phillips theorem the isomorphism class of $\mathcal{O}_E$ is completely determined by the $K$-theory of $A$ together with the class of the unit (since $\mathcal{O}_E$ is $KK$-equivalent to $A$).

Many examples of Cuntz-Pimsner algebras found in the literature arise from Hilbert bimodules which are finitely generated and projective; in such cases the left action must consist entirely of compact operators. Our examples do not fall in this class; in fact, the left action has trivial intersection with the compacts. And this has some interesting consequences: $\mathcal{O}_E \cong \mathcal{T}_E$ (see [Pm, Corollary 3.14]) and the natural embedding $A \hookrightarrow \mathcal{O}_E$ induces a $KK$-equivalence (see [Pm, Corollary 4.5]).

In §1 we review some basic facts concerning the construction of $\mathcal{T}_E$ as operators on the Fock space of $E$ and the gauge action $\lambda : T \to \text{Aut}(\mathcal{T}_E)$. We assume that the left action of $A$ does not meet the compacts $\mathcal{K}(E)$ and identify $\mathcal{O}_E$ with $\mathcal{T}_E$. The fixed point algebra $\mathcal{F}_E$, the analogue of the AF-core of a Cuntz-Krieger algebra, contains a canonical descending sequence of essential ideals indexed by $\mathbb{N}$ with trivial intersection. The crossed product $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ has a similar collection of essential ideals indexed by $\mathbb{Z}$ on which the dual group of automorphisms acts in a natural way. By Takesaki-Takai duality

$$\mathcal{O}_E \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_\lambda \mathbb{Z};$$

hence, much of the structure of $\mathcal{O}_E$ is revealed through an analysis of the double crossed product.


1991 Mathematics Subject Classification. Primary 46L05; Secondary 46L55.

Key words and phrases. C*-algebra, Hilbert bimodule, simple, purely infinite.

Research partially supported by NSF grant DMS-9706982.
In §2 we show that if $E$ is the Hilbert bimodule over $A$ associated to a representation as described above, then for every nonzero positive element $d \in \mathcal{O}_E$ there is a $z \in \mathcal{O}_E$ so that $z^*dz = 1$; it follows that $\mathcal{O}_E$ is simple and purely infinite (see Theorem 2.8). The proof of this proceeds through a sequence of lemmas and is patterned on the proof of [Re, Theorem 2.1], which is in turn based on a key lemma of Kishimoto (see [Ks, Lemma 3.2]). Our argument uses the version of this lemma found in [OP3, Lemma 7.1] and this requires that we show that the Connes spectrum of the dual action is full (this is also an ingredient in the proof of simplicity found in [DPW]). We invoke a version of a key lemma of Rørdam for crossed products by $\mathbb{Z}$ which arise from automorphisms with full Connes spectrum. The fact that $\mathcal{O}_E$ embeds equivariantly into $(\mathcal{O}_E \times_A \mathbb{T}) \rtimes_\alpha \mathbb{Z}$ allows us to apply this lemma to $\mathcal{O}_E$. In §3 we use the Kirchberg-Phillips theorem to collect some consequences of this theorem as indicated above and discuss certain connections with reduced (amalgamated) free products.

We fix some notation and terminology. Given a $C^*$-algebra $B$ we let $\widehat{B}$ denote its spectrum, that is, the collection of irreducible representations modulo unitary equivalence endowed with the Jacobson topology (see [Pd, §4.1]). If $I$ is an ideal in a $C^*$-algebra $B$, then every irreducible representation of $I$ extends uniquely to an irreducible representation of $B$. This allows one to identify $I$ with an open subset of $\widehat{B}$, the complement of which consists of the classes of irreducible representations which vanish on $I$. Given a $^*$-automorphism $\beta$ of a $C^*$-algebra $B$, let $\Gamma(\beta)$ denote the Connes spectrum of $\beta$ (see [O, Co] or [Pd, §8.8]); recall that $\Gamma(\beta) = \bigcap_H \text{Sp}(\beta|_H)$ where the intersection is taken over all nonzero $\beta$-invariant hereditary subalgebras $H$. A $C^*$-algebra is said to be purely infinite if every nonzero hereditary subalgebra contains an infinite projection. I wish to thank D. Shlyakhtenko for certain helpful remarks relating to material in §3.

1. Preliminaries

We review some basic facts concerning Cuntz-Pimsner algebras; we shall be mainly interested in those which arise from bimodules for which the left action has trivial intersection with the compacts (see Remark 1.3). Let $A$ be a $C^*$-algebra.

**Definition 1.1.** (see [L, pp 2–4], [Ka, pp 134, 135] and [Ri1, Def. 2.1]) Let $E$ be a right $A$-module. Then $E$ is said to be a (right) pre-Hilbert $A$-module if it is equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ which satisfies the following conditions for all $\xi, \eta, \zeta \in E$, $s, t \in \mathbb{C}$, and $a \in A$:

i. $\langle \xi, s\eta + t\zeta \rangle_A = s\langle \xi, \eta \rangle_A + t\langle \xi, \zeta \rangle_A$

ii. $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$

iii. $\langle \eta, \xi A \rangle_A = \langle \xi, \eta \rangle_A^*$

iv. $\langle \xi, \xi \rangle_A \geq 0$ and $\langle \xi, \xi \rangle_A = 0$ only if $\xi = 0$.

$E$ is said to be a (right) Hilbert $A$-module if it is complete in the norm: $\|\xi\| = \|\langle \xi, \xi \rangle_A\|^{1/2}$.

Let $E$ be a Hilbert $A$-module. Then $E$ is said to be full if the span of the values of the inner product is dense. The collection of bounded adjointable operators on $E$, $\mathcal{L}(E)$, is a $C^*$-algebra. The closure of the span of operators of the form $\theta_{\xi, \eta}$ for $\xi, \eta \in E$ (where $\theta_{\xi, \eta}(z) = \xi(\eta z)$ for $z \in E$) forms an essential ideal in $\mathcal{L}(E)$ which is denoted $\mathcal{K}(E)$. A Hilbert space is a Hilbert module over $\mathbb{C}$.

**Definition 1.2.** Let $E$ be a Hilbert $A$-module and $\varphi : A \to \mathcal{L}(E)$ be an injective $^*$-homomorphism. Then the pair $(E, \varphi)$ is said to be Hilbert bimodule over $A$ (or Hilbert $A$-bimodule).

Pimsner defines the Cuntz-Pimsner algebra $\mathcal{O}_E$ as a quotient of the analogue of the Toeplitz algebra, $\mathcal{T}_E$, generated by creation operators on the Fock space of $E$ (see [Pm]). The injectivity
of $\varphi$ is not really necessary (see [Pm, Remark 1.2(1)]). We will henceforth assume that $E$ is full (see [Pm, Remark 1.2(3)]).

The Fock space of $E$ is the Hilbert $A$-module

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^\otimes n$$

where $E^\otimes 0 = A$, $E^\otimes 1 = E$ and for $n > 1$, $E^\otimes n$ is the $n$-fold tensor product:

$$E^\otimes n = E \otimes_A \cdots \otimes_A E.$$

The tensor product used here is called the inner tensor product by Lance (see [L, p 41]), but note Lance uses different notation; see also Theorem 5.9 of [Ri1]). Observe that $\mathcal{E}_+$ is also a Hilbert $A$-bimodule with left action defined by $\varphi_+(a) b = ab$ for $a, b \in A = E^\otimes 0$ and

$$\varphi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \varphi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a \in A$ and $\xi_1 \otimes \cdots \otimes \xi_n \in E^\otimes n$.

Then $T_E \subset \mathcal{L}(\mathcal{E}_+)$ is the $C^*$-algebra generated by the creation operators $T_\xi$ for $\xi \in E$ where $T_\xi(a) = \xi a$ and

$$T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

Note that $T_\xi^* T_\eta = \varphi_+((\xi, \eta)_A)$ for $\xi, \eta \in E$. Since $E$ is full, $\varphi_+(A) \subset T_E$; let $\iota : A \hookrightarrow T_E$ denote the embedding. One may also define $T_\xi$ for $\xi \in E^\otimes n$, in an analogous manner and we have $T_\xi^* T_\eta = \iota((\xi, \eta)_A)$ for $\xi, \eta \in E^\otimes n$.

There is an embedding $\iota_n : \mathcal{K}(E^\otimes n) \hookrightarrow T_E$ (identify $\mathcal{K}(E^\otimes 0)$ with $A$), given for $n > 0$ by

$$\iota_n(\theta_{\xi, \eta}) = T_\xi^* T_\eta^*$$

for $\xi, \eta \in E^\otimes n$. Note that such operators preserve the grading of $\mathcal{E}_+$ and that there is an embedding $\mathcal{K}(E^\otimes n) \hookrightarrow \mathcal{L}(E^\otimes m)$ for $m \geq n$. Let $C_n$ denote the $C^*$-subalgebra of $T_E$ generated by operators of the form $T_\xi T_\eta^*$ for $\xi, \eta \in E^\otimes k$ with $k \leq n$ (by convention $C_0 = \iota(A)$). Then the $C_n$ form an ascending family of $C^*$-subalgebras.

**Remark 1.3.** With notation as above the natural map $C_n \to \mathcal{L}(E^\otimes m)$ is an embedding for $m \geq n$. Suppose $\varphi(A) \cap \mathcal{K}(E) = \{0\}$; then by [Pm, Corollary 3.14] $T_E \cong \mathcal{O}_E$ and the inclusion $A \hookrightarrow \mathcal{O}_E$ induces a $KK$-equivalence (see [Pm, Corollary 4.5]). Under the isomorphism of $T_E$ with $\mathcal{O}_E$, $\cup_n C_n$ is mapped to $\mathcal{F}_E$, the analog of the AF core of a Cuntz-Krieger algebra.

For the remainder of this section we shall tacitly assume that $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and identify $T_E$ with $\mathcal{O}_E$.

**Proposition 1.4.** For each $n \in \mathbb{N}$ the $C^*$-subalgebra, $J_n$, generated by $\iota_n(\mathcal{K}(E^\otimes k))$ for $k \geq n$ is an essential ideal in $\mathcal{F}_E$. We obtain a descending sequence of ideals

$$J_0 \supset J_1 \supset J_2 \supset \cdots$$

with $J_0 = \mathcal{F}_E$ and $\cap_n J_n = \{0\}$. Furthermore, $J_n / J_{n+1} \cong \mathcal{K}(E^\otimes n)$ (thus $J_n / J_{n+1}$ is strong Morita equivalent to $A$) and the restriction of the quotient map yields an isomorphism $C_n \cong \mathcal{F}_E / J_{n+1}$.

**Proof.** Given $n \in \mathbb{N}$ it is clear that $J_n$ is an ideal (see [Pm, Definition 2.1]). To see that $J_n$ is essential it suffices to show that for every $m$ and nonzero element $c \in C_m$ there is an element $d \in \mathcal{K}(E^\otimes k)$ for some $k \geq n$ such that $c d k(d) \neq 0$. Let $k$ be an integer with $k \geq \max(m, n)$; since the map from $C_m$ to $\mathcal{L}(E^\otimes k)$ is an embedding for $k \geq m$, $c \xi \neq 0$ for some $\xi \in E^\otimes k$. Then $c T_\xi T_\xi^* \neq 0$ and we take $d = c T_\xi$.

The $J_n$ form a descending sequence of ideals by construction. Since $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and $\mathcal{K}(E) \hookrightarrow \mathcal{L}(E^\otimes k)$ is nondegenerate for $k \geq 1$, the image of $A$ in $\mathcal{L}(E^\otimes k)$ has trivial intersection with $\mathcal{K}(E^\otimes k)$ for $k \geq 1$; it follows that

$$\iota_n(\mathcal{K}(E^\otimes m)) \cap \iota_n(\mathcal{K}(E^\otimes n)) = \{0\}$$

and, hence, $C_m \cap J_n = \{0\}$ for $m < n$. Thus, $\cap_n J_n = \{0\}$, for $\mathcal{F}_E$ is the inductive limit of the $C_n$. Further, for each $n$ we have

$$J_n = \iota_n(\mathcal{K}(E^\otimes n)) + J_{n+1} \quad \text{and} \quad \iota_n(\mathcal{K}(E^\otimes n)) \cap J_{n+1} = \{0\}.$$
it follows that \( J_n / J_{n+1} \cong K(E^{\otimes n}) \). Finally, since

\[ \mathcal{F}_E = C_n + J_{n+1} \quad \text{and} \quad C_n \cap J_{n+1} = \{0\}, \]
we have \( C_n \cong \mathcal{F}_E / J_{n+1} \).

There is a strongly continuous action

\[ \lambda : \mathbb{T} \to \text{Aut} (O_E) \]

such that \( \lambda (t) f \mathcal{T}_n = t \mathcal{T}_n \). The fixed point algebra under this action is \( \mathcal{F}_E \) and we have a faithful conditional expectation \( P_E : O_E \to \mathcal{F}_E \) given by

\[ P_E (x) = \int \lambda_t (x) \, dt. \]

Consider the spectral subspaces of \( O_E \) under this action: for \( n \in \mathbb{Z} \)

\[ (O_E)_n = \{ x \in O_E : \lambda_t (x) = t^n x \ \text{for all} \ t \in \mathbb{T} \}. \]

**Remark 1.5.** Note that \((O_E)_n \) is the closure of the span of elements of the form \( \mathcal{T}_n^* \mathcal{T}_n \) where \( \xi \in E^{\otimes k} \) and \( \eta \in E^{\otimes l} \) with \( n = k - l \). For \( n \geq 0 \) and \( x \in (O_E)_n \), we have \( x^* x \in \mathcal{F}_E \) and \( xx^* \in J_n \). We may regard \((O_E)_n \) as a \( J_n - \mathcal{F}_E \) equivalence bimodule (or \( J_n - \mathcal{F}_E \) imprimitivity bimodule; see [Ri1, Def. 6.10]). Hence, \( J_n \) is strongly Morita equivalent to \( \mathcal{F}_E \) for each \( n \geq 0 \) (see [Ri2, Def. 1.1], [L, p. 74]). If we regard \((O_E)_1 \) as a Hilbert \( \mathcal{F}_E \)-bimodule, we have

\[ E \otimes_A \mathcal{F}_E \cong (O_E)_1, \]

where the isomorphism is implemented by the map \( \xi \otimes a \mapsto T_\xi a \) (the Hilbert \( \mathcal{F}_E \)-module \( E \otimes_A \mathcal{F}_E \) is denoted \( E_\infty \) in [Pin, §2]). The crossed product \( O_E \rtimes \mathbb{T} \) may be identified with the closure of the subalgebra of \( O_E \otimes K(\ell^2 (\mathbb{Z})) \) consisting of finite sums of the form

\[ \sum_{ij} x_{ij} \otimes e_{ij} \]

where \( e_{ij} \) are the standard rank one partial isometries in \( K(\ell^2 (\mathbb{Z})) \) and \( x_{ij} \in (O_E)_{j-i} \).

Let \( \hat{\lambda} : \mathbb{Z} \to \text{Aut} (O_E \rtimes \mathbb{T}) \) denote the dual automorphism group.

**Proposition 1.6.** There is an embedding \( \epsilon : \mathcal{F}_E \hookrightarrow O_E \rtimes \mathbb{T} \) onto a corner and a collection of essential ideals \( \{I_n\}_{n \in \mathbb{Z}} \) in \( O_E \rtimes \mathbb{T} \) satisfying the following conditions:

i. For all \( n \in \mathbb{Z} \), \( \mathcal{F}_E \) is strongly Morita equivalent to \( I_n / I_{n+1} \) and \( A \) is strongly Morita equivalent to \( I_n / I_{n+1} \).
ii. For all \( n \geq 0 \), \( \epsilon (I_n) = \epsilon (1) I_n \epsilon (1) \).
iii. \( I_n \subseteq I_m \) if \( m \leq n \).
iv. \( \cap_n I_n = \{0\} \)

v. \( \frac{I_n}{I_m} = O_E \rtimes \mathbb{T} \)

vi. \( \hat{\lambda} (I_n) = I_{n+k} \)

**Proof.** We use the identification of \( O_E \rtimes \mathbb{T} \) with a \( C^* \)-subalgebra of \( O_E \otimes K(\ell^2 (\mathbb{Z})) \) given in Remark 1.5. For each \( n \) let \( I_n \) be the ideal generated by \( p_n = 1 \otimes e_{mn} \). Since \( \mathcal{F}_E = (O_E)_0 \), it follows that \( \mathcal{F}_E \) is isomorphic to the corner determined by \( p_n \) and thus is strongly Morita equivalent to \( I_n \). The desired embedding \( \epsilon : \mathcal{F}_E \hookrightarrow O_E \rtimes \mathbb{T} \) is given by \( \epsilon (a) = a \otimes e_{00} \).

Given an element of the form \( a_{mn} = x_{mn} \otimes e_{mn} \) in \( O_E \rtimes \mathbb{T} \) with \( m \leq n \), we have

\[ a_{mn}^* a_{mn} = x_{mn}^* x_{mn} \otimes e_{mn} \quad \text{and} \quad a_{mn}^* a_{mn}^* = x_{mn} x_{mn}^* \otimes e_{mn}. \]

with \( x_{mn} x_{mn}^* \in J_{n-m} \); since \( p_n \) may be expressed as a finite sum of elements of the form \( a_{mn}^* a_{mn} \), it follows that \( I_n \subseteq I_m \) and that

\[ p_m I_n p_m = J_{n-m} \otimes e_{mn}. \]

Moreover, \( I_n \) is essential in \( I_m \), since \( J_{n-m} \) is an essential ideal in \( \mathcal{F}_E \) (by Proposition 1.4). Since \( q_n = \sum_{i=-n}^{n} p_i \in I_n \) and \( \{q_n\}_n \) forms an approximate identity, we have \( \cup_n I_n = O_E \rtimes \mathbb{T} \). Thus
Let $A$ be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of $A$ on a separable nontrivial Hilbert space $\mathfrak{H}$; since $\pi$ is nondegenerate we have $\pi(1) = 1$.

**Proposition 2.1.** With $A$ and $\pi : A \to \mathcal{L}(\mathfrak{H})$ as above, 
$$E = \mathfrak{H} \otimes_C A$$

is a full Hilbert bimodule over $A$ under the operations
$$(\xi \otimes a, \eta \otimes b)_A = (\xi, \eta)a^*b, \quad \varphi(a)(\xi \otimes b) = \pi(a)\xi \otimes b$$
for all $\xi, \eta \in \mathfrak{H}$ and $a, b \in A$. Moreover, if $\pi(A) \cap K(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap K(E) = \{0\}$ and $O_E \cong T_E$.

**Proof.** Note that $E = \mathfrak{H} \otimes_C A$ is the tensor product of the Hilbert $A$-$C$-bimodule $\mathfrak{H}$ and the Hilbert $C$-$A$-bimodule $A$ as defined by Rieffel in [Ri1, Theorem 5.9] (see also [L, p 41]). The natural map from $\mathcal{L}(\mathfrak{H})$ to $\mathcal{L}(E) = \mathcal{L}(\mathfrak{H} \otimes_C A)$ induces an embedding $\mathcal{L}(\mathfrak{H})/K(\mathfrak{H}) \hookrightarrow \mathcal{L}(E)/K(E)$ (since $K(\mathfrak{H})$ is mapped into $K(E)$ and the Calkin algebra $\mathcal{L}(\mathfrak{H})/k(\mathfrak{H})$ is simple). Hence, if $\pi(A) \cap K(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap K(E) = \{0\}$. The last assertion, $O_E \cong T_E$, follows by [Pm, Corollary 3.14].

Henceforth, we assume that $\pi(A) \cap K(\mathfrak{H}) = \{0\}$ and identify $O_E$ with $T_E$. The aim of this section is to show that $O_E$ is simple and purely infinite. Simplicity may be proven directly by invoking [Sc2, Theorem 3.9]: if $A$ is unital and $E$ is full, then $O_E$ is simple if and only if $E$ is minimal and nonperiodic. Lemma 2.3 would then be a consequence of [OP1, Theorem 6.5]. We follow a more indirect route patterned on the proof of [Ro, Theorem 2.1]; this will also show that $O_E$ is purely infinite.

**Remark 2.2.** With $E = \mathfrak{H} \otimes_C A$ as above, we have $E^\otimes_n \cong \mathfrak{H}^\otimes_n \otimes_C A$ via the map
$$(\xi_1 \otimes a_1) \otimes (\xi_2 \otimes a_2) \otimes \cdots \otimes (\xi_n \otimes a_n) \mapsto (\xi_1 \otimes \pi(a_1)\xi_2 \otimes \cdots \otimes \pi(a_{n-1})\xi_n) \otimes a_n.$$ If $\sigma : A \to \mathcal{L}(\mathfrak{R})$ is a nondegenerate representation of $A$ on a Hilbert space $\mathfrak{R}$, then
$$E \otimes_A \mathfrak{R} \cong \mathfrak{H} \otimes_C A \otimes_A \mathfrak{R} \cong \mathfrak{H} \otimes_C \mathfrak{R}$$
and, hence,
$$E^\otimes_n \otimes_A \mathfrak{R} \cong E^\otimes_{n-1} \otimes_A E \otimes_A \mathfrak{R} \cong E^\otimes_{n-1} \otimes_A \mathfrak{H} \otimes_C \mathfrak{R}.$$ Recall that the action of $\mathcal{F}_E$ on Fock space preserves the natural grading. Let $\tilde{\sigma}_n$ denote the representation of $\mathcal{F}_E$ on $E^\otimes_n \otimes_A \mathfrak{R}$ given by left action on $E^\otimes_n$. Then the restriction of $\tilde{\sigma}_n$ to $C_{n-1}$ is faithful: indeed, this follows from the facts that the natural map
$$\mathcal{L}(E^\otimes_{n-1}) \to \mathcal{L}(E^\otimes_{n-1} \otimes_A \mathfrak{H} \otimes_C \mathfrak{R}) \cong \mathcal{L}(E^\otimes_n \otimes_A \mathfrak{R})$$
is an embedding (since $\pi$ is faithful) and that $\tilde{\sigma}_n|_{K(E^\otimes_{n-1})}$ factors through $\mathcal{L}(E^\otimes_{n-1})$. Note that $\tilde{\sigma}_n$ is equivalent to the representation of $\mathcal{F}_E$ obtained from $\sigma$ as follows: use the strong Morita equivalence between $A$ and $J_n/J_{n+1}$ to obtain a representation of $J_n/J_{n+1}$ and extend this to a representation of $\mathcal{F}_E$. Since the restriction of $\tilde{\sigma}_n$ to $C_{n-1}$ is faithful, let $\tilde{\sigma}_n \subset C_n$ (see Proposition 1.4). It follows that the closure of a point in $\tilde{J}_n - \tilde{J}_{n+1}$ contains the complement of $\tilde{J}_n$. A similar assertion holds for $O_E \rtimes_\lambda \mathbb{T}$: for any $n \in \mathbb{Z}$ the closure of a point in $\tilde{I}_n - \tilde{I}_{n+1}$ contains the complement of $\tilde{I}_n$.

**Lemma 2.3.** With $A$ and $E$ as above, $\Gamma(\lambda_1) = \mathbb{T}$ where $\lambda$ is the dual action of $\mathbb{Z}$ on $O_E \rtimes_\lambda \mathbb{T}$. 


Proof. By [OP2, Theorem 4.6] it suffices to find a dense invariant subset of \((\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^\sim\) on which \(\hat{\lambda}_t^*\) acts freely. That is, we must find an irreducible representation \(\sigma\) of \(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}\) such that, 
\[
\{[\sigma \circ \hat{\lambda}_n] : n \in \mathbb{Z}\},
\]
the orbit of the unitary equivalence class of \(\sigma\) under \(\hat{\lambda}_t^*\), is dense in \((\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^\sim\) and \([\sigma \circ \hat{\lambda}_m] \neq [\sigma \circ \hat{\lambda}_n]\) if \(m \neq n\). Let \(\sigma_0\) be an irreducible representation of \(A\) and use the strong Morita equivalence between \(A\) and \(I_0/I_1\) to obtain an irreducible representation \(\sigma'\) of \(I_0/I_1\). Then \(\sigma\), the extension of \(\sigma'\) to \(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}\), is also irreducible. The classes \([\sigma \circ \hat{\lambda}_n]\) are distinct, for if \(m < n\), \(\sigma \circ \hat{\lambda}_m\) vanishes on \(I_n\). Moreover, for each \(n \in \mathbb{Z}\) the closure of \([\sigma \circ \hat{\lambda}_n]\) in \((\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^\sim\) includes the classes of all irreducible representations which vanish on \(I_n\) (since \([\sigma \circ \hat{\lambda}_n]\) \(\in I_n - I_{n+1}\); see Remark 2.2). Hence, \([\sigma \circ \hat{\lambda}_n] : n \in \mathbb{Z}\) is dense in \((\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^\sim\). □

Using Takesaki-Takai duality we show below that a C*-algebra \(D\) equipped with an action \(\alpha\) of \(\mathbb{T}\) may be embedded invariantly as a corner in \((D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}\). This fact is related to Rosenberg’s observation that the fixed point algebra under a compact group action embeds as a corner in the crossed product (see [Ro]).

**Proposition 2.4.** Given a unital C*-algebra \(D\) and a strongly continuous action \(\alpha : \mathbb{T} \to \text{Aut}(D)\), there is an isomorphism \(\psi\) of \(D\) onto a full corner of \((D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}\) which is equivariant in the sense that \(\widehat{\alpha} \circ \psi = \psi \circ \alpha_t\) for all \(t \in \mathbb{T}\). Moreover, \(\psi(1) \in (D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}\).

Proof. By Takesaki-Takai duality [Pd, 7.9.3] there is an isomorphism
\[
\gamma : D \otimes K(L^2(\mathbb{T})) \cong (D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z},
\]
which is equivariant with respect to \(\alpha \otimes \text{Ad} \rho \) and \(\widehat{\alpha}\) (where \(\rho\) is the right regular representation of \(\mathbb{T}\) on \(L^2(\mathbb{T})\)). The desired embedding is obtained by finding an \(\text{Ad} \rho\) invariant minimal projection \(p \in K(L^2(\mathbb{T}))\) (cf. [Ro]): set \(\psi(d) = \gamma(d \otimes p)\) for \(d \in D\). Since \(\psi\) is equivariant, \(\psi(1)\) is in the fixed point algebra of \(\hat{\alpha}\); hence, \(\psi(1) \in (D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}\). □

The following lemma is adapted from [Ro, Lemma 2.4]; the proof is patterned on Rørdam’s but we substitute [OP3, Lemma 7.1] for [Ks, Lemma 3.2].

**Lemma 2.5.** Let \(B\) be a C*-algebra and let \(\beta\) be an automorphism of \(B\) such that \(\Gamma(\beta) = \mathbb{T}\) and let \(P\) denote the canonical conditional expectation from \(B \rtimes_{\beta} \mathbb{Z}\) to \(B\). Then for every positive element \(y \in B \rtimes_{\beta} \mathbb{Z}\) and \(\epsilon > 0\) there are positive elements \(x, b \in B\) such that
\[
\|b\| > \|P(y)\| - \epsilon, \quad \|x\| \leq 1 \text{ and } \|xyx - b\| < \epsilon.
\]

If \(y\) is in the corner determined by a projection \(p \in B\), then \(x, b\) may also be chosen to be in the corner.

Proof. As in the proof of [Ro, Lemma 2.4] we may assume (by perturbing \(y\) if necessary) that \(y\) is of the form
\[
y = y_{-n} u^{-n} + \cdots + y_{-1} u^{-1} + y_0 + y_1 u + \cdots + y_n u^n
\]
for some \(n\) where \(y_i \in B\) and \(u\) is the canonical unitary in \(B \rtimes_{\beta} \mathbb{Z}\) implementing the automorphism \(\beta\); note that \(y_0 = P(y)\) is positive. By [OP3, Theorem 10.4] \(\beta^k\) is properly outer for all \(k \neq 0\). Hence, by [OP3, Lemma 7.1] there is a positive element \(x\) with \(\|x\| = 1\) such that
\[
\|xy_0 x\| > \|y_0\| - \epsilon, \quad \text{and} \quad \|xy_k u^k x\| = \|xy_k \beta^k (x)\| < \epsilon/2n
\]
for \(0 < |k| \leq n\). Set \(b = xy_0 x\); then a straightforward calculation yields \(\|xyx - b\| < \epsilon\). We now verify the last assertion. Suppose that \(y\) is in the corner determined by a projection \(p \in B\); we may again assume that \(y\) is of the above form. Since \(P\) is a conditional expectation onto \(B\), \(y_0 = P(y)\) is also in the corner determined by \(p\). In the proof of [OP3, Lemma 7.1] the positive element \(x\) is constructed in the hereditary subalgebra determined by \(y_0\); hence we may assume that \(x\) and therefore also \(b = xy_0 x\) lies in the same corner. □
Recall that $C_n$ is the $C^*$-subalgebra of $\mathcal{F}_E$ generated by operators of the form $T_\xi T_\eta^*$ for $\xi, \eta \in E^{\otimes k}$ with $k \leq n$ and that they form an ascending family of $C^*$-subalgebras with dense union. The subspace $E^{\otimes n}$ is left invariant by $C_n$ and one has an embedding $C_n \hookrightarrow \mathcal{L}(E^{\otimes n})$.

**Lemma 2.6.** Given a positive element $c \in C_n$ and $\varepsilon > 0$, there is $\xi \in E^{\otimes n}$ with $\|\xi\| = 1$ such that $T_\xi^* c T_\xi \in C_0$ and $\|T_\xi^* c T_\xi\| > \|c\| - \varepsilon$.

**Proof.** The first assertion follows from a straightforward calculation: given $c \in C_n$ and $\xi \in E^{\otimes n}$, then $c \xi \in E^{\otimes n}$ and

$$T_\xi^* c T_\xi = T_\xi^* T_\xi \in \iota(\langle \xi, c \xi \rangle) \subset C_0.$$ 

The second assertion follows from the embedding $C_n \hookrightarrow \mathcal{L}(E^{\otimes n})$ and the fact

$$\|d\| = \sup\{\|\langle \xi, d \xi \rangle\| : \xi \in E^{\otimes n}, \|\xi\| = 1\}$$

for $d \in \mathcal{L}(E^{\otimes n})$ positive. \hfill $\Box$

**Lemma 2.7.** Given a positive element $a \in A$ and $\varepsilon > 0$ with $\|a\| > \varepsilon$, there is $\eta \in E$ with $\|\eta\| \leq (\|a\| - \varepsilon)^{-1/2}$ such that $T_\eta^* \iota(a) T_\eta = 1$.

**Proof.** Let $f$ be a continuous nonzero real-valued function supported on the interval $[\|a\| - \varepsilon, \|a\|]$ and choose a vector $\xi \in \pi(f(a))$ such that $\langle \xi, \pi(a) \xi \rangle = 1$; we have

$$\|\xi\|^2 \leq \|\xi, \pi(a) \xi\| = 1.$$ 

Then $\eta = \xi \otimes 1 \in E$ satisfies the desired conditions. \hfill $\Box$

It will now follow that $O_E$ is simple and purely infinite (cf. proof of [Rø, Theorem 2.1]).

**Theorem 2.8.** For every nonzero positive element $d \in O_E$ there is a $z \in O_E$ so that $z^* dz = 1$. Hence, $O_E$ is simple and purely infinite.

**Proof.** Let $d \in O_E$ be a nonzero positive element and choose $\varepsilon$ so that $0 < \varepsilon < \|P(d)\|/4$. By Proposition 2.4 there is a $\mathbb{T}$-equivariant isomorphism $\psi$ from $O_E$ onto a corner of $(O_E \rtimes_\lambda \mathbb{T}) \rtimes_\lambda \mathbb{Z}$ determined by a projection $p \in O_E \rtimes_\lambda \mathbb{T}$. We now apply Lemma 2.5 to the element $y = \psi(d)$ and the automorphism $\beta = \hat{\lambda}_1$ (note $\Gamma(\hat{\lambda}_1) = \mathbb{T}$ by Lemma 2.3). We identify $O_E$ with the corner determined by $p$; note that under this identification $\mathcal{F}_E$ is identified with $p(O_E \rtimes_\lambda \mathbb{T})p$. There are then positive elements $x, b \in F_E$ so that

$$\|b\| > \|P(d)\| - \varepsilon, \|x\| \leq 1 \quad \text{and} \quad \|xdx - b\| < \varepsilon.$$ 

Since $\cup n C_n$ is dense in $F_E$ we may assume that $b \in C_n$ for some $n$. Hence, by Lemma 2.6 there is $\xi \in E^{\otimes n}$ with $\|\xi\| = 1$ such that

$$T_\xi^* b T_\xi \in C_0 \quad \text{and} \quad \|T_\xi^* b T_\xi\| > \|b\| - \varepsilon.$$ 

Let $a$ denote the unique element of $A$ such that $\iota(a) = T_\xi^* b T_\xi$; then $\|a\| > \|P(d)\| - 2\varepsilon$ and

$$\|T_\xi^* xdx T_\xi - \iota(a)\| = \|T_\xi^* xdx - \iota(a)\| < \varepsilon.$$ 

By Lemma 2.7 there is $\eta \in E$ such that $T_\eta^* \iota(a) T_\eta = 1$ and

$$\|\eta\| \leq (\|a\| - \varepsilon)^{-1/2} < (\|P(d)\| - 3\varepsilon)^{-1/2} < \varepsilon^{-1/2}.$$ 

It follows that

$$\|T_\eta^* T_\xi^* xdx T_\xi T_\eta - 1\| = \|T_\eta^* (T_\xi^* xdx T_\xi - \iota(a)) T_\eta\| \leq \|T_\xi^* xdx T_\xi - \iota(a)\| (\varepsilon^{-1/2})^2 < 1.$$ 

Therefore, $c = T_\eta^* T_\xi^* xdx T_\xi T_\eta$ is an invertible positive element and we take $z = xT_\xi T_\eta c^{-1/2}$. \hfill $\Box$
3. Applications and concluding remarks

We collect some applications of the above theorem and consider certain connections with the theory of reduced (amalgamated) free product C*-algebras. First we consider criteria under which the Kirchberg-Phillips Theorem applies (see [Kr, Theorem C], [Ph, Corollary 4.2.2]).

**Theorem 3.1.** Let $A$ be a separable nuclear unital C*-algebra which belongs to the bootstrap class to which the Kirchberg-Phillips Theorem applies (see [RS]; [Kr, Theorem C], [Ph, Corollary 4.2.2]). Then $\mathcal{O}_E$ is a unital Kirchberg algebra (simple, purely infinite, separable and nuclear) which belongs to the bootstrap class. Hence, the Kirchberg-Phillips Theorem applies and the isomorphism class of $\mathcal{O}_E$ only depends on $(K_0(A), [1_A])$ and not on the choice of representation $\pi$.

**Proof.** First note that $\mathcal{O}_E$ is simple and purely infinite by Theorem 2.8. If $A$ is nuclear, then the argument given in the proof of [DS, Theorem 2.1] shows that $\mathcal{O}_E$ must also be nuclear (alternatively, the nuclearity of $\mathcal{O}_E$ follows from the structural results discussed in §1). Hence, $\mathcal{O}_E$ is a unital Kirchberg algebra. Recall that the inclusion $A \hookrightarrow \mathcal{O}_E$ defines a $KK$-equivalence (see [Pm, Corollary 4.5]) which induces a unit-preserving isomorphism $K_0(A) \cong K_0(\mathcal{O}_E)$. Hence, if $A$ is in the bootstrap class, then $\mathcal{O}_E$ is also. Therefore, the Kirchberg-Phillips Theorem applies and the isomorphism class of $\mathcal{O}_E$ only depends on $(K_0(A), [1_A])$. $\square$

Let $X$ be a second countable compact space, let $\mu$ be a nonatomic Borel measure with full support and let

$$\pi : C(X) \rightarrow \mathcal{L}(L^2(X, \mu))$$

be the representation given by multiplication of functions. Then $\pi$ is faithful and

$$\pi(C(X)) \cap K(L^2(X, \mu)) = \{0\}.$$ 

Hence, we may apply the above theorem with $A = C(X)$ and $\mathcal{H} = L^2(X, \mu)$.

**Corollary 3.2.** Let $X$ and $\mu$ be as above. Then

$$E = L^2(X, \mu) \otimes \mathbb{C} C(X)$$

is a Hilbert bimodule over $C(X)$ and $\mathcal{O}_E$ is a unital Kirchberg algebra. The embedding $C(X) \hookrightarrow \mathcal{O}_E$ induces a (unit preserving) $KK$-equivalence. Hence, the isomorphism class of $\mathcal{O}_E$ only depends on $(K_0(C(X)), [1_{C(X)}])$ (and not on $\mu$); moreover, if $X$ is contractible, then $\mathcal{O}_E \cong \mathcal{O}_\infty$.

The following proposition is Theorem 5.6 of [L] (see also [Ka, Theorem 3]); Lance calls this the Kasparov-Stinespring-Gelfand-Naimark-Segal construction.

**Proposition 3.3.** Let $B$ and $C$ be C*-algebras, let $F$ be a Hilbert $C$-module and let $f : B \rightarrow \mathcal{L}(F)$ be a completely positive map. Then there is a Hilbert $C$-module $E_f$, a $*$-homomorphism $\varphi_f : B \rightarrow \mathcal{L}(E_f)$ and an element $v_f \in \mathcal{L}(F, E_f)$ such that $f(b) = v_f^* \varphi_f(b) v_f$ and $\varphi_f(B) v_f F$ is dense in $E_f$.

I am grateful to D. Shlyakhtenko for the following observation. Let $T$ denote the “usual” Toeplitz algebra (i.e. $T_E$ where $E$ is the 1-dimensional Hilbert bimodule over $\mathbb{C}$) and let $g$ denote the vacuum state on $T$.

**Proposition 3.4.** Let $A$ be a separable unital C*-algebra and let $\pi : A \rightarrow \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi$ has a cyclic vector $\xi \in \mathcal{H}$. Let $f$ denote the vector state $\langle \xi, \pi(\cdot) \xi \rangle$ and let $\tilde{f}$ denote the corresponding completely positive map from $A$ to $\mathcal{L}(A)$ (given by $\tilde{f}(a) = f(a) 1$). Then $E = E_{\tilde{f}} \cong \mathcal{H} \otimes A$ and $T_E$ may be realized as a reduced free product (see [A, V]):

$$(T_E, h) \simeq (A, f) * (T, g)$$

for some state $h$ on $T_E$. 

Proof. This follows from [Sh, Theorem 2.3, Corollary 2.5].

As a result of this observation part (at least) of Corollary 3.2 follows from the existing literature on reduced free products. The simplicity follows from a theorem of Dykema [Dy, Theorem 2]. Criteria for when reduced free products are purely infinite have been found by Choda, Dykema and Rørdam in a series of papers [DR1, DR2, DC]; but none seem to apply generally to the case considered in the corollary.

A theorem of Speicher (see [Sp]) on reduced amalgamated free products (see [V, §5]) and Toeplitz algebras associated to Hilbert bimodules yields a curious stability property of the algebras we have been considering. The following is the version given in [BDS, Theorem 2.4].

**Proposition 3.5.** Suppose that \( E_1 \) and \( E_2 \) are full Hilbert bimodules over the C*-algebra \( A \). Then

\[
    \mathcal{T}_{E_1 \oplus E_2} = \mathcal{T}_{E_1} *_A \mathcal{T}_{E_2}.
\]

We obtain the following corollary.

**Corollary 3.6.** Let \( A \) be a separable nuclear unital C*-algebra which belongs to the bootstrap class to which the UCT applies (see [RS]) and let \( \pi : A \to \mathcal{L}(\mathfrak{H}) \) be a faithful representation of \( A \) on a separable Hilbert space \( \mathfrak{H} \) such that \( \pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\} \). Let \( E \) be the Hilbert bimodule \( \mathfrak{H} \otimes_C A \). Then

\[
    \mathcal{O}_E \cong \mathcal{O}_E *_A \mathcal{O}_E.
\]

Proof. Observe that \( E \oplus E = (\mathfrak{H} \oplus \mathfrak{H}) \otimes_C A \). Since \( \pi \oplus \pi : A \to \mathcal{L}(\mathfrak{H} \oplus \mathfrak{H}) \) is a faithful representation and \( (\pi \oplus \pi)(A) \cap \mathcal{K}(\mathfrak{H} \oplus \mathfrak{H}) = \{0\} \), the result follows follows from Theorem 3.1 and the above proposition.

**References**


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