Diagonals in Fell algebras, an interim report.

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Introduction.

A Fell algebra is a C^*-algebra A which satisfies Fell's condition; this is equivalent to requiring that abelian elements generate. In this case A is almost a continuous trace algebra but $\widehat{A}$ need not be Hausdorff. Such algebras arise naturally in the study of certain dynamical systems. We prove:

- An abelian C^*-subalgebra B of a Fell algebra A is a diagonal iff it satisfies the extension property.
- Up to Rieffel-Morita equivalence (RME) each such A contains a diagonal.
- The twists arising from RME Fell algebras containing diagonals are "equivalent".
- We define a complete invariant for Fell algebras: $\delta(A) \in \mathrm{Br}(X)$ where A is a Fell algebra with $\widehat{A} = X$ (see [DD63]).

Fell algebras and abelian elements.

Definitions (see [Ped79]): Let A be a C^*-algebra.

- An element $a \in A_+$ is said to be *abelian* if $\overline{aAa}$ is abelian.
- We say that A is *type* I_0 if it is generated by abelian elements.

It is easy to show that a C^*-algebra A is type I_0 iff it satisfies Fell's condition: for every $x \in \widehat{A}$, there is an element $p \in A_+$ such that $\pi(p)$ is a rank 1 projection for every irreducible representation π with $[\pi]$ in a neighborhood of x. Continuous trace algebras are Fell.

Note, $C^*(\mathbb{Z} \rtimes \mathbb{Z}_2)$ is a Fell algebra with non-Hausdorff spectrum.

Suppose A is separable; A is Fell iff there are ideals J_n such that

- J_n is RME to an abelian algebra,
- the J_n generate A as an ideal.

Hence, if A is Fell, then $\widehat{A}$ is locally Hausdorff.

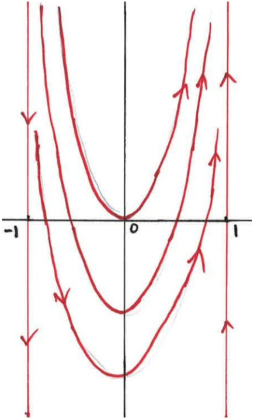
Example.

Dynamical systems which give rise to Fell algebras have been characterized in [anH01].

There is a free action of $\mathbb{R}$ on the strip $X := [-1, 1] \times \mathbb{R}$ with non-Hausdorff quotient.

Using this characterization one sees that the crossed product $C_0(X) \rtimes \mathbb{R}$ is Fell.

Its spectrum is the quotient of two copies of $[0, 1)$ identified along $(0, 1)$.



Astrid's sketch.

Extension property.

Let B be a C^*-subalgebra of A. Following Archbold et al (see [ABG82]) we say that B has the *extension property* relative to A if

- pure states of B extends uniquely,
- B contains an approximate unit for A.

The extension property passes to quotients.

If B is an abelian C^*-subalgebra with the extension property, then

- B is a maximal abelian subalgebra of A,
- there is a unique conditional expectation $P : A \to B$,
- for each pure state $x \in \widehat{B}$, its unique extension is given by $x \circ P$ (which is necessarily pure).

Diagonals.

Let B be an abelian C^*-subalgebra of A. An element $n \in A$ is said to be a *normalizer* of B if $nBn^*, n^*Bn \subset B$. Such an n is said to be *free* if $n^2 = 0$. The collection of free normalizers is denoted $N_f(B)$.

Definition (see [Kum86]): B is said to be a diagonal in A if

- there is a faithful conditional expectation $P : A \to B$,
- B contains an approximate unit for A,
- $\operatorname{span} N_f(B)$ is dense in $\ker P$.

A complete invariant for the diagonal pair (A, B) is given by an extension of groupoids with common unit space $\widehat{B}$,

$$\widehat{B} \times \mathbb{T} \to \Gamma \to R,$$

sometimes refered to as the *twist invariant*.

The spectral map Ψ.

Let B be an abelian C^*-subalgebra of A which has the extension property relative to A. Given $x \in \widehat{B}$, we let $(\pi_x, \mathcal{H}_x, \xi_x)$ denote the GNS triple associated to the pure state $x \circ P$.

We define the *spectral map* $\Psi : \widehat{B} \to \widehat{A}$ by $\Psi(x) = [\pi_x]$.

Lemma: If A is liminary, Ψ is surjective.

Proof (sketch): Let π be an irreducible representation of A. Then $\pi(B)$ is a masa in $\pi(A) = \mathcal{K}(\mathcal{H})$. Hence, there exist $\{x_n\} \subset \widehat{B}$ and a family of orthogonal rank one projections $\{p_n\}$ such that

$$\pi(b) = \sum_n x_n(b) p_n \quad \text{for } b \in B.$$

Therefore, $\Psi^{-1}([\pi]) = \{x_n\}$ and Ψ is onto.

Structure Theorem.

Let B be an abelian C^*-subalgebra of A which has the extension property relative to A.

If A is Fell, then it is liminary and the previous lemma holds.

But more is true (cf. [Kum85]).

Theorem: Suppose that A is a Fell algebra. Then

- Ψ is a local homeomorphism,
- B is a diagonal in A,
- $R = R(\Psi)$.

The theorem fails if A is only assumed to be liminary.

Note that diagonals automatically satisfy the extension property.

Using abelian elements to construct a diagonal.

Let A be a separable Fell algebra. Then A is generated by a countable family of abelian elements $\{a_n\}$. We construct a diagonal in a C^*-algebra RME to A as follows.

We let $\mathcal{K}$ denote the compacts and let $\{e_{mn}\}$ be a family of matrix units. Set

$$a := \sum_n 2^{-n} a_n \otimes e_{nn}$$

and $C := \overline{a(A \otimes \mathcal{K})a}$. Then C is RME to A.

Moreover, the C^*-algebra D generated by elements of the form

$$a_n b a_n \otimes e_{nn} \qquad \text{where } b \in A$$

is a diagonal in C.

Twist invariants of RME Fell algebras.

There were choices made in the above diagonal construction. What, if anything, can one use to derive an invariant?

Proposition: Let A_1 and A_2 be Fell algebras which are RME and let L be a linking algebra containing A_1 and A_2. Suppose that B_i is a diagonal in A_i for $i = 1, 2$. Then $B_1 \oplus B_2$ is a diagonal in L.

Corollary: With assumptions as above, the associated twist invariants

$$\widehat{B}_i \times \mathbb{T} \to \Gamma_i \to R_i$$

are equivalent in the sense of Renault (see [Ren85], [Kum86]).

Corollary: The equivalence class of the twist obtained from the diagonal construction is independent of the choices made and only depends on the RME class of A.

An analog of the Dixmier-Douady invariant.

Let (A, B) be a diagonal pair where A is Fell algebra with $\widehat{A} = X$.
We define the Brauer group $\mathrm{Br}(X) := H^2(R(\Psi), \mathcal{S})$ where Ψ is the spectral map associated to the diagonal pair (A, B) and $\mathcal{S}$ is the sheaf of germs of continuous $\mathbb{T}$-valued functions. The group $\mathrm{Br}(X)$ does not depend on the choices made.
Let $\mathrm{Tw}(G)$ denote the group of twists over an étale groupoid G. There is a natural map $\partial^1 : \mathrm{Tw}(G) \to H^2(G, \mathcal{S})$.
With (A, B) as above, let Γ denote the twist invariant and define

$$\delta(A) = \partial^1([\Gamma]) \in H^2(R(\Psi), \mathcal{S}) = \mathrm{Br}(X).$$

Two twists arising from RME equivalent diagonal pairs map to the same element of $\mathrm{Br}(X)$.

A liminary C^*-algebra

We construct an example of a liminary algebra with an abelian subalgebra which has the extension property but is not a diagonal.
Set

$$C := \{f \in C([0,1], M_2) : f(0) \in \mathbb{C}I\},$$

and let D be the subalgebra of C consisting of functions f such that each $f(t)$ is a diagonal matrix.
Then C is liminary and D is a diagonal subalgebra which therefore satisfies the extension property with respect to C.
For $t > 0$, let

$$u_t := \begin{pmatrix} \cos(1/t) & \sin(1/t) \\ -\sin(1/t) & \cos(1/t) \end{pmatrix}.$$

Define $\alpha \in \mathrm{Aut}(C)$ by

$$\alpha(f)(t) = \begin{cases} u_t f(t) u_t^* & \text{if } t > 0 \\ f(0) & \text{if } t = 0. \end{cases}$$

Let $A := M_2(C)$ and let B be the abelian subalgebra of matrices of the following form

$$\begin{pmatrix} d_1 & 0 \\ 0 & \alpha(d_2) \end{pmatrix}, \qquad \text{where } \ d_1, d_2 \in D.$$

Then A is liminary and B the extension property relative to A, but B is not a diagonal subalgebra of A.

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Thanks!

Any questions?

Thanks, Arlan, for organizing this gathering!