On the simplicity of twisted $k$-graph $C^*$-algebras

Preliminary report of work in progress

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GPOTS14, Kansas State University,
Manhattan, 27 May 2014
Introduction

$C^*$-algebras of higher-rank graphs (or $k$-graphs) were introduced in [KP00] as generalizations of graph $C^*$-algebra which model $C^*$-algebras arising from certain group actions on buildings discovered by Robertson and Steger. If a $k$-graph $\Lambda$ satisfies mild hypotheses, there is a path groupoid $\mathcal{G}_\Lambda$ such that $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$. Using this isomorphism it was shown that $C^*(\Lambda)$ is simple iff $\Lambda$ is aperiodic and cofinal.

Given a $k$-graph $\Lambda$ and a $\mathbb{T}$-valued 2-cocycle $c$, one may form the twisted $k$-graph $C^*$-algebra $C^*(\Lambda, c)$ (see [KPS]). Examples include all noncommutative tori and crossed products of Cuntz algebras by quasifree automorphisms.

Our goal is to characterize the simplicity of $C^*(\Lambda, c)$. Sufficient conditions were given in (see [SWW]).

This talk is based on joint work with David Pask and Aidan Sims of the University of Wollongong.
Let $k \in \mathbb{N} := \{0, 1, 2, \ldots \}$.

**Definition (see [KP00])**

Let $\Lambda$ be a countable small category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. Then $(\Lambda, d)$ is a *k-graph* if it satisfies the factorization property:

For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu \nu$, $d(\mu) = m$ and $d(\nu) = n$.

Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \text{Obj} (\Lambda)$, the set of *vertices*. An element $\lambda \in \Lambda^{e_i}$ is called an *edge*.

We assume throughout that $\Lambda$ is *row-finite* and *source-free*, that is for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$, $v\Lambda^n := r^{-1}(v) \cap \Lambda^n$ is finite and nonempty.
Let $\Lambda$ be a $k$-graph.

- If $k = 0$, then $d$ is trivial and $\Lambda$ is just a set.
- If $k = 1$, then $\Lambda$ is the path category of a directed graph.
- If $k \geq 2$, think of $\Lambda$ as generated by $k$ graphs of different colors that share the same set of vertices $\Lambda^0$.

The $k$-graph $T_k := \mathbb{N}^k$ may be regarded as the $k$-graph analog of a torus.

Let $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$ be the $k$-graph with structure maps

\[
s(m, n) = n \quad r(m, n) = m \quad d(m, n) = n - m \quad (\ell, m)(m, n) = (\ell, n)
\]
The path groupoid

The infinite path space $\Lambda^\infty$ is the set of $k$-graph morphisms $x : \Omega_k \to \Lambda$. The shift map: for $q \in \mathbb{N}^k$ define $\sigma^q : \Lambda^\infty \to \Lambda^\infty$ by

$$\sigma^q(x)(m, n) = x(m + q, n + q) \quad \text{for } (m, n) \in \Omega_k.$$

We define the path groupoid $\mathcal{G}_\Lambda \subset \Lambda^\infty \times \mathbb{Z} \times \Lambda^\infty$ by

$$\mathcal{G}_\Lambda := \{ (x, m - n, y) : \sigma^m(x) = \sigma^n(y) \text{ for some } m, n \in \mathbb{N}^k \}.$$

The unit space is identified with $\Lambda^\infty$ via the map $x \mapsto (x, 0, x)$.

$\Lambda$ is cofinal if for every $v \in \Lambda^0$ and $x \in \Lambda^\infty$, there are $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n, n)$ and $r(\lambda) = v$. If $\Lambda$ is cofinal, $\mathcal{G}_\Lambda$ is minimal.

For $v \in \Lambda^0$ the local periodicity group at $v$ $P_\Lambda(v)$ is the set of all $m - n \in \mathbb{Z}^k$ such that $m, n \in \mathbb{N}^k$ and $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\Lambda^\infty$.

$\Lambda$ is aperiodic if $P_\Lambda(v) = 0$ for all $v \in \Lambda^0$. If $\Lambda$ is aperiodic, $\mathcal{G}_\Lambda$ is topologically principal, that is, points with trivial isotropy are dense.
The categorical cohomology, $H^\ast_{\text{cat}}(\Lambda, A)$, is the usual cocycle cohomology for groupoids (see [R]) extended to small categories.

Consider the 2-cocycle $c \in Z^2_{\text{cat}}(\Lambda, A)$, that is, a map $c : \Lambda \ast \Lambda \to A$ such that for any composable triple $(\lambda_1, \lambda_2, \lambda_3)$ we have

$$c(\lambda_1, \lambda_2) + c(\lambda_1\lambda_2, \lambda_3) = c(\lambda_1, \lambda_2\lambda_3) + c(\lambda_2, \lambda_3)$$

and $c$ is a 2-coboundary if there is $b : \Lambda \to A$ such that

$$c(\lambda_1, \lambda_2) = b(\lambda_1) - b(\lambda_1\lambda_2) + b(\lambda_2).$$

$H^2_{\text{cat}}(\Lambda, A)$ is the quotient group (2-cocycles modulo 2-coboundaries).

There is a homomorphism $H^2_{\text{cat}}(\Lambda, A) \to H^2(\mathcal{G}_{\Lambda}, A)$ induced by a map $c \mapsto \sigma_c$ (see [KPS, §6]).
The $C^*$-algebra $C^*(\Lambda, c)$

Definition (see [KPS, §5])

For $c \in \mathbb{Z}^2_{\text{cat}}(\Lambda, \mathbb{T})$ let $C^*(\Lambda, c)$ be the universal $C^*$-algebra generated by the set $\{t_\lambda : \lambda \in \Lambda\}$ satisfying:

1. $\{t_v : v \in \Lambda^0\}$ is a family of orthogonal projections.
2. For $\lambda \in \Lambda$, $t_{s(\lambda)} = t_\lambda^* t_\lambda$.
3. If $s(\lambda) = r(\mu)$, then $t_\lambda t_\mu = c(\lambda, \mu) t_{\lambda \mu}$.
4. For $v \in \Lambda^0$, $n \in \mathbb{N}^k$

$$t_v = \sum_{\lambda \in v \Lambda^n} t_\lambda t_\lambda^*.$$

Remark: If $c$ and $c'$ are cohomologous, then $C^*(\Lambda, c) \cong C^*(\Lambda, c')$.

Theorem (see [KPS, §7])

Let $c \in \mathbb{Z}^2_{\text{cat}}(\Lambda, \mathbb{T})$ and let $\sigma_c \in \mathbb{Z}^2(\mathcal{G}_\Lambda, \mathbb{T})$ be as above. Then

$$C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c).$$
Structure of $C^* (\Lambda, c)$ when $\Lambda$ is cofinal

If $C^* (\Lambda, c)$ is simple, then $\Lambda$ is cofinal but not necessarily aperiodic.

Suppose $\Lambda$ is cofinal. Then $P_\Lambda := P_\Lambda (v)$ does not depend on $v \in \Lambda^0$ and there is a short exact sequence of étale groupoids

$$\Lambda^\infty \times P_\Lambda \overset{i}{\to} \mathcal{G}_\Lambda \to \mathcal{H}_\Lambda$$

where $\mathcal{H}_\Lambda$ is minimal and topologically principal (see [KPSS]).

The cohomology class of $i^*_x (\sigma_c)$ in $H^2 (P_\Lambda, \mathbb{T})$ is independent of $x$.

Moreover, there is $\sigma \in Z^2 (\mathcal{G}_\Lambda, \mathbb{T})$ and $\omega \in Z^2 (P_\Lambda, \mathbb{T})$ such that $[\sigma] = [\sigma_c]$ and $i^*_x (\sigma) = \omega$ for all $x \in \Lambda^\infty$. It follows that

$$C^* (\Lambda^\infty \times P_\Lambda, i^*(\sigma)) \cong C_0 (\Lambda^\infty) \otimes C^* (P_\Lambda, \omega).$$

Theorem

There is a Fell bundle $\mathcal{B}_\Lambda^c$ over $\mathcal{H}_\Lambda$ such that $\mathcal{B}_\Lambda^c \big|_{\Lambda^\infty} \cong \Lambda^\infty \times C^* (P_\Lambda, \omega)$ and

$$C^* (\Lambda, c) \cong C^* (\mathcal{H}_\Lambda; \mathcal{B}_\Lambda^c).$$

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Simplicity itself

Let $Z_\omega := \{ q \in P_\Lambda \mid \omega(p, q)\omega(q, p) = 1 \}$ then by [OPT] we have

$$\text{Prim } C^*(P_\Lambda, \omega) \cong \hat{Z}_\omega.$$ 

By [IW, §2] there is an action of $H_\Lambda$ on

$$\text{Prim } C_0(\Lambda^\infty) \otimes C^*(P_\Lambda, \omega) = \Lambda^\infty \times \hat{Z}_\omega.$$ 

The action is determined by a cocycle $\tilde{c} \in Z^1(H_\Lambda, \hat{Z}_\omega)$.

**Theorem**

*Suppose $\Lambda$ is cofinal. Then $C^*(\Lambda, c)$ is simple iff the action of $H_\Lambda$ on $\Lambda^\infty \times \hat{Z}_\omega$ is minimal.*

If $Z_\omega = 0$, then $C^*(\Lambda, c)$ is simple. But this condition is not necessary as the following example shows.
An example

In this example $\Lambda$ is cofinal and $\omega$ is trivial, but $C^*(\Lambda, c)$ is simple.

Let $B_2$ be the 1-graph with 1 vertex and 2 edges and let $T_1 = \mathbb{N}$ be the 1-graph with 1 vertex and 1 edge.

Let $\Lambda := B_2 \times T_1$ and $c((\mu, m), (\nu, n)) := z^{nd(\mu)}$ where $z := e^{2\pi i \theta}$ and $\theta$ is irrational.

Since $P_\Lambda \cong \mathbb{Z}$, $\omega$ is a coboundary. Thus $Z_\omega = P_\Lambda$.

We have $G_\Lambda \cong H_\Lambda \times \mathbb{Z}$ and $\Lambda^\infty \cong B_2^\infty = \{(x_1, x_2, \ldots) \mid x_i = a_1 \text{ or } a_2\}$.

Moreover, $\tilde{c}(x, n, y) = z^{-n}$ and hence $C^*(\Lambda, c)$ is simple.
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