Twisted topological graph $C^*$-algebras

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In his thesis Hui Li showed how Katsura’s construction of the $C^*$-algebra of a topological graph $E$ may be twisted by a Hermitian line bundle $L$ over the edge space $E^1$. The correspondence defining the algebra is obtained as the completion of the compactly supported continuous sections of $L$. In joint work with Li we prove that the resulting $C^*$-algebra is isomorphic to a twisted groupoid $C^*$-algebra. We use the Renault-Deaconu groupoid of the topological graph with Yeend’s boundary path space as its unit space.

Throughout all topological spaces and groupoids are assumed to be locally compact, Hausdorff and second countable. A groupoid is said to be étale if the source map is a local homeomorphism.
Let $A$ be a $C^*$-algebra. A right $A$-module $X$ is said to be a right Hilbert $A$-module if it is endowed with an $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ such that for all $x, y \in X$ and $a \in A$

i. $\langle x, y \rangle_A = \langle y, x \rangle^*_A$,

ii. $\langle x, ya \rangle_A = \langle x, y \rangle_A a$,

iii. $\langle x, x \rangle_A \geq 0$ and $\langle x, x \rangle_A = 0$ iff $x = 0$,

and it is complete in the norm defined by $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

If in addition there is a $*$-homomorphism $\varphi : A \to \mathcal{L}(X)$ the pair $(X, \varphi)$ (or simply $X$) is called a $C^*$-correspondence over $A$.

Pimsner’s definition of the Cuntz-Pimsner algebra of a correspondence was reformulated by Katsura.
The Cuntz-Pimsner algebra

Let $B$ be a $C^*$-algebra, let $\psi : X \to B$ be a linear map and let $\pi : A \to B$ be a $*$-homomorphism. Then $(\psi, \pi)$ is said to be a representation of $(X, A)$ if for all $a \in A$ and $x, y \in X$

i. $\psi(\varphi(a)x) = \pi(a)\psi(x)$,

ii. $\pi(\langle x, y \rangle_A) = \psi(x)^* \psi(y)$,

and it follows that $\psi(xa) = \psi(x)\pi(a)$ and $\|\psi(x)\| \leq \|x\|$.

Note that there is a natural map $\psi^{(1)} : \mathcal{K}(X) \to B$ such that $\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$ for all $x, y \in X$.

The representation $(\psi, \pi)$ is said to be covariant if $\pi(a) = \psi^{(1)}(\varphi(a))$ for all $a \in \varphi^{-1}(\mathcal{K}(X)) \cap \ker \varphi^\perp$.

The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the codomain of a universal covariant representation $(\psi_X, \pi_X)$. So if $(\psi, \pi)$ is any covariant representation with codomain $B$ there is a unique $*$-homomorphism $\eta : \mathcal{O}_X \to B$ such that $\psi = \eta \circ \psi_X$ and $\pi = \eta \circ \pi_X$. 
An earlier version of topological graph was introduced by Deaconu.

**Definition (Katsura)**

A quadruple $E = (E^0, E^1, r, s)$ is called a topological graph if $E^0$, $E^1$ are topological spaces, $r : E^1 \to E^0$ is a continuous map, and $s : E^1 \to E^0$ is a local homeomorphism.

There is natural $C^*$-correspondence $X = X(E)$ over $A := C_0(E^0)$ obtained as the completion of $C_c(E^1)$.

The structure maps on the dense bimodule $C_c(E^1)$ are defined for $f \in C_0(E^0)$ and $g, h \in C_c(E^1)$ by

i. $(gf)(t) = g(t)f(s(t))$ for all $t \in E^1$;

ii. $\langle g, h \rangle_A(u) = \sum_{u = s(t)} g(t)h(t)$ for all $u \in E^0$;

iii. $(\varphi(f)g)(t) = f(r(t))g(t)$ for all $t \in E^1$.

The $C^*$-algebra of the topological graph is defined to be $\mathcal{O}_X(E)$. 

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Twisted Topological Graphs

Let $E$ be a topological graph and let $p : L \rightarrow E^1$ be a Hermitian line bundle with a continuous family of conjugate linear inner products $\langle \cdot, \cdot \rangle : L_t \times L_t \rightarrow \mathbb{C}$ for $t \in E^1$.

Note that for $f, g \in C_c(E^1; L)$ (compactly supported continuous sections of $L$) the map $t \mapsto \langle f(t), g(t) \rangle$ lies in $C_c(E^1)$.

The twisted $C^*$-correspondence $X(E, L)$ over $A := C_0(E^0)$ is obtained as the completion of the dense bimodule $C_c(E^1; L)$.

The structure maps on $C_c(E^1; L)$ are as above except that the $A$-valued inner product is given by

$$\langle g, h \rangle_A(u) = \sum_{u = s(t)} \langle g(t), h(t) \rangle \quad \text{for } g, h \in C_c(E^1; L), u \in E^0.$$ 

The twisted topological graph algebra is defined to be $O_{X(E, L)}$.

We have an isomorphism of $C_0(E^0)$-correspondences

$$X(E, L) \cong C_0(E^1; L) \otimes_{C_0(E^1)} X(E).$$
It is well known that the $\mathcal{C}^*$-algebra of a directed graph is isomorphic to the $\mathcal{C}^*$-algebra of the associated path groupoid which may be identified with the Renault-Deaconu groupoid of the shift map on its path space.

Yeend has shown that given a topological graph $E$ one may similarly construct a boundary path space $\partial E$ and an étale groupoid $\Gamma$ with $\Gamma^0 = \partial E$ such that $\mathcal{C}^*(\Gamma) \cong \mathcal{O}_{X(E)}$.

The fact that Yeend’s groupoid $\Gamma$ is the Renault-Deaconu groupoid of the partially defined shift map on $\partial E$ was implicit in his work. We use ideas of Webster to simplify the description of $\partial E$.

Katsura also proved $\mathcal{O}_{X(E)}$ is isomorphic to a Renault-Deaconu groupoid associated to the topological graph in some cases.

Our goal is to prove a twisted version of Yeend’s result.
A partial local homeomorphism on $T$ is a local homeomorphism $\sigma : \text{dom}(\sigma) \rightarrow \text{ran}(\sigma)$ where $\text{dom}(\sigma)$ and $\text{ran}(\sigma)$ are open in $T$.

**Definition (Renault)**

The Renault-Deaconu groupoid $\Gamma(T, \sigma) \subset T \times \mathbb{Z} \times T$ is given by:

$$\{(t_1, k_1 - k_2, t_2) : k_1, k_2 \geq 0, t_i \in \text{dom}(\sigma^{k_i}), \sigma^{k_1}(t_1) = \sigma^{k_2}(t_2)\}.$$ 

We identify $T \cong \Gamma(T, \sigma)^0 := \{(t, 0, t) : t \in T\}$ by $t \mapsto (t, 0, t)$.

We have $\Gamma(T, \sigma)^2 := \{((t_1, n, t_2), (t_2, m, t_3))\}$ and structure maps

$$r(t_1, n, t_2) := t_1, \quad (t_1, n, t_2)^{-1} := (t_2, -n, t_1),$$

$$s(t_1, n, t_2) := t_2, \quad (t_1, n, t_2)(t_2, m, t_3) := (t_1, n + m, t_3).$$

The topology is generated by sets of the form $U(U_1, U_2, k_1, k_2)$ where $U_i \subset \text{dom}(\sigma^{k_i})$ is open, $T^{k_i}$ is injective on $U_i$ and $U(U_1, U_2, k_1, k_2) := \{(t_1, k_1 - k_2, t_2) : t_i \in U_i, \sigma^{k_1}(t_1) = \sigma^{k_2}(t_2)\}$. 

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**Definition**

Let $\Gamma$ and $\Lambda$ be groupoids with common unit space $\Gamma^0$ and s.t. $\Gamma$ is étale. Then $\Lambda$ is said to be a *topological twist* over $\Gamma$ if there is a sequence of unit space fixing groupoid homomorphisms

$$\mathbb{T} \times \Gamma^0 \xrightarrow{i} \Lambda \xrightarrow{p} \Gamma$$

such that

1. $i$ is a homeomorphism onto $p^{-1}(\Gamma^0)$;
2. $\Lambda$ is a principal $\mathbb{T}$-bundle over $\Gamma$ with bundle map $p$;
3. $\lambda i(z, s(\lambda))\lambda^{-1} = i(z, r(\lambda))$, for all $z \in \mathbb{T}$, and all $\lambda \in \Lambda$.

The twisted groupoid $C^*$-algebra $C^* (\Gamma; \Lambda)$ is defined to be the closure of $\{ f \in C_c(\Lambda) : f(z\lambda) = zf(\lambda), z \in \mathbb{T}, \lambda \in \Lambda \}$ in $C^*(\Lambda)$. 

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Let $X(E, L)$ be the correspondence associated to the twisted topological graph $(E, L)$. Let $SL$ denote the associated circle bundle over $E^1$ (consisting of the unit vectors of $L$).

There is a partial local homeomorphism $\sigma$ on $\partial E$ and a continuous surjection $q : \text{dom } \sigma \to E^1$. Moreover there is an embedding

$$j : \text{dom } \sigma \to \Gamma(\partial E, \sigma)$$

given by $j(t) := (t, 1, \sigma(t))$.

**Theorem**

There is a unique twist $\Lambda$ over $\Gamma(\partial E, \sigma)$ such that $j^*(\Lambda) \cong q^*(SL)$. Moreover

$$\mathcal{O}_{X(E, L)} \cong C^*(\Gamma(\partial E, \sigma); \Lambda).$$


Thanks!

Questions?
Quantum Heisenberg Manifolds

In ’89 Rieffel introduced quantum Heisenberg manifolds $D_{\mu,\nu}^c$, where $\mu, \nu \in \mathbb{R}$ and $c \in \mathbb{N}$ as key examples of his deformation quantization theory.

Work of Abadie, Eilers and Exel suggested in ’98 that the $D_{\mu,\nu}^c$ are isomorphic to Cuntz-Pimsner algebras associated to twisted topological graphs with $E^0 = E^1 = \mathbb{T}^2$, where $s = \text{id}$ and $r$ is translation determined by $\mu, \nu$, and $L$ is a Hermitian line bundle determined by the integer $c$.

Abadie showed that $K_0(D_{\mu,\nu}^c) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_c$ and $K_1(D_{\mu,\nu}^c) \cong \mathbb{Z}^3$. Consequently $c$ is an isomorphism invariant of $D_{\mu,\nu}^c \cong \mathcal{O}_X(E,L)$.

Work of Kang, K. and Packer in ’16 shows that $D_{\mu,\nu}^c$ is a twisted groupoid $C^*$-algebra.
Let $E$ be a topological graph. For $n \geq 2$, define

$$E^n := \left\{ \mu = (\mu_1, \ldots, \mu_n) \in \prod_{i=1}^{n} E^1 : s(\mu_i) = r(\mu_{i+1}), \ i < n \right\}$$

endowed with the subspace topology of the product space $\prod_{i=1}^{n} E^1$. Define the *finite-path space* $E^* := \bigsqcup_{n=0}^{\infty} E^n$ with the disjoint union topology. Define the *infinite-path space*

$$E^\infty := \left\{ \mu \in \prod_{i=1}^{\infty} E^1 : s(\mu_i) = r(\mu_{i+1}), \ i = 1, 2, \ldots \right\}.$$
Define

1. \( E_{\text{sce}}^0 := E^0 \setminus \overline{r(E^1)}. \)
2. \( E_{\text{fin}}^0 := \{ v \in E^0 : \text{there exists an open neighborhood } N \text{ of } v \text{ such that } r^{-1}(N) \text{ is compact} \}. \)
3. \( E_{\text{rg}}^0 := E_{\text{fin}}^0 \setminus E_{\text{sce}}^0. \)
4. \( E_{\text{sg}}^0 := E^0 \setminus E_{\text{rg}}^0. \)

Let \( E \) be a topological graph. Define the boundary path space by

\[ \partial E := E^\infty \cup \{ \mu \in E^* : s(\mu) \in E_{\text{sg}}^0 \}. \]
Proposition (Yeend)

Let $E$ be a topological graph. There is a locally compact Hausdorff topology on the boundary paths space $\partial E$ with the following characterization of convergent sequences in $\partial E$.

For a sequence $(\mu^{(n)})_{n=1}^{\infty}$ in $\partial E$, $\mu \in \partial E$, we have $\mu^{(n)} \to \mu$ if and only if

1. $r(\mu^{(n)}) \to r(\mu)$;
2. for $1 \leq i \leq |\mu|$ with $i \neq \infty$, there exists $N \geq 1$ such that $|\mu^{(n)}| \geq i$ whenever $n \geq N$ and $(\mu_1^{(n)} \cdots \mu_i^{(n)})_{n \geq N} \to \mu_1 \cdots \mu_i$;
3. if $|\mu| < \infty$, then for any compact set $K \subset E^1$, the set $\{n : |\mu^{(n)}| > |\mu|, \text{ and } \mu^{(n)}_{|\mu|+1} \in K\}$ is finite.