The Ext class of an approximately inner automorphism, II

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Abstract

Let $A$ be a simple unital AT algebra of real rank zero and $\text{Inn}(A)$ the group of inner automorphisms of $A$. In the previous paper we have shown that the natural map of the group $\overline{\text{Inn}}(A)$ of approximately inner automorphisms into

$$\text{Ext}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A))$$

is surjective; the kernel of this map includes the subgroup of automorphisms which are homotopic to $\text{Inn}(A)$. In this paper we consider the quotient of $\text{Inn}(A)$ by the smaller normal subgroup $\text{AInn}(A)$ which consists of asymptotically inner automorphisms and describe it as

$$\text{OrderExt}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A)),$$

where $\text{OrderExt}(K_1(A), K_0(A))$ is a kind of extension group which takes into account the fact that $K_0(A)$ is an ordered group and has the usual $\text{Ext}$ as a quotient.  \footnote{1991 Mathematics Subject Classification: Primary 46L40; Secondary 46L80, 46L35. Research supported by NSF grant DMS-9706982

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1 Introduction

An automorphism $\alpha$ of a unital $C^*$-algebra $A$ is called inner if there is a unitary $u \in A$ such that $\alpha(a) = Ad_u(a) = uau^*$, $a \in A$. We denote by $\text{Inn}(A)$ the group of inner automorphisms of $A$, which is a normal subgroup of the group $\text{Aut}(A)$ of all automorphisms of $A$. The topology on $\text{Aut}(A)$ is determined by pointwise convergence on $A$. The closure $\overline{\text{Inn}}(A)$ of $\text{Inn}(A)$ in $\text{Aut}(A)$ is, by definition, the group of approximately inner automorphisms.

There are two distinguished normal subgroups of $\overline{\text{Inn}}(A)$ containing $\text{Inn}(A)$. One is the group $\text{HInn}(A)$ of automorphisms which are homotopic to $\text{Inn}(A)$, i.e., $\alpha \in \text{HInn}(A)$ if and only if there is a continuous map $\alpha : [0,1] \rightarrow \overline{\text{Inn}}(A)$ such that

$$\alpha_0 \in \text{Inn}(A), \quad \alpha_1 = \alpha.$$ 

The other is the group $\text{AInn}(A)$ of asymptotically inner automorphisms, i.e., $\alpha \in \text{AInn}(A)$ if and only if there is a continuous map $\alpha : [0,1] \rightarrow \overline{\text{Inn}}(A)$ and a continuous map $u : [0,1] \rightarrow U(A)$ with $U(A)$ the unitary group of $A$ such that

$$\alpha_t = Ad_{u_t} \text{ for } t \in [0,1], \quad \alpha_1 = \alpha.$$ 

It is easy to show that they are indeed normal subgroups and that

$$\text{Inn}(A) \subset \text{AInn}(A) \subset \text{HInn}(A) \subset \overline{\text{Inn}}(A).$$

In this paper we describe the quotient

$$\overline{\text{Inn}}(A)/\text{AInn}(A)$$

in terms of $K$-theoretic data when $A$ is a simple unital AT algebra of real rank zero.

Recall that a unital $C^*$-algebra $A$ is said to be a unital AT algebra if it is expressible as the inductive limit of $T$ algebras, i.e., finite direct sums of matrix algebras over $C(T)$, with unital embeddings. Note that a unital AT algebra $A$ is stably finite and we denote by $T_A$ the convex set of tracial states of $A$.  \footnote{1991 Mathematics Subject Classification: Primary 46L40; Secondary 46L80, 46L35. Research supported by NSF grant DMS-9706982

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Let $A$ be a simple unital $\text{AT}$ algebra of real rank zero and $\alpha \in \text{Inn}(A)$. (In this case $\alpha \in \text{Aut}(A)$ belongs to $\text{Inn}(A)$ if and only if $\alpha_* = \text{id}$ on $K_*(A)$ [7].) The mapping torus of $\alpha$ is the $C^*$-algebra:

$$M \alpha = \{ x \in C[0,1] \otimes A; \; \alpha(x(0)) = x(1) \}.$$ 

The suspension of $A$, $SA$, is identified with the ideal of $M \alpha$:

$$SA = \{ x \in C[0,1] \otimes A; \; x(0) = 0 = x(1) \}.$$ 

From the short exact sequence:

$$0 \rightarrow SA \rightarrow M \alpha \rightarrow A \rightarrow 0,$$

one obtains the usual six-term exact sequence in K-theory, which, since $\alpha K$ for $\tau$ induces a surjective homomorphism:

$$i_{\tau} \rightarrow 1$$

for sequences:

$$0 \rightarrow K_i(A) \rightarrow K_{i+1}(M \alpha) \rightarrow K_{i+1}(A) \rightarrow 0$$

for $i = 0, 1$, where $K_{i+1}(SA)$ has been identified with $K_i(A)$. Let $\eta_i(\alpha)$ denote the class of this sequence in $\text{Ext}(K_{i+1}, K_i(A))$ and let $\eta$ denote the map of $\text{Inn}(A)$ into

$$\oplus_{i=0}^1 \text{Ext}(K_{i+1}(A), K_i(A))$$

defined by $\alpha \mapsto (\eta_0(\alpha), \eta_1(\alpha))$, which is a group homomorphism. (By using $KK$ theory and the universal coefficient theorem [13], $\eta(\alpha)$ is also described as $KK(\alpha) - KK(\text{id})$.) In the previous paper [11] we showed that $\eta$ induces a surjective homomorphism:

$$\text{Inn}(A)/\text{HInn}(A) \rightarrow \text{Ext}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A)).$$

To state our main result of this paper we proceed to describe a natural map $R_\alpha$ of $K_1(M \alpha)$ into $\text{Aff}(T_A)$, which is the real Banach space of affine continuous functions on the compact tracial state space $T_A$ of $A$. Note that, since we assume that $A$ has real rank zero, $T_A$ is isomorphic to the state space of ($K_0(A)$, [1]). If $u \in M \alpha$ is a unitary given as a piecewise smooth function of $[0,1]$ into $A$, then $R_\alpha([u])$ is defined by

$$R_\alpha([u])(t) = \frac{1}{2\pi i} \int_0^1 \tau(\dot{u}(t)u(t)^*)dt$$

for $\tau \in T_A$. The map $R_\alpha$ is a group homomorphism of $K_1(M \alpha)$ into $\text{Aff}(T_A)$ and extends the natural map $D$ of $K_0(A)$ into $\text{Aff}(T_A)$ when $K_0(A)$ is regarded as a subgroup of $K_1(M \alpha)$.

We take the set of pairs $(E, R)$ where $E$ is an abelian group such that

$$0 \rightarrow K_0(A) \rightarrow E \xrightarrow{\iota} K_1(A) \rightarrow 0$$

and $R$ is a homomorphism:

$$R : E \rightarrow \text{Aff}(T_A)$$

such that $R \circ \iota = D$. We can form a group $\text{OrderExt}(K_1(A), K_0(A))$ from this set in much the same way as we do $\text{Ext}(K_1(A), K_0(A))$ from the set of $E$ alone. From the previous paragraph we can associate $\tilde{\eta}_0(\alpha) \in \text{OrderExt}(K_1(A), K_0(A))$ with each $\alpha \in \text{Inn}(A)$ and show that $\tilde{\eta}_0$ is a homomorphism. Our main result is

$$\text{Inn}(A)/\text{AInn}(A) \cong \text{OrderExt}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A))$$

where the isomorphism is induced by the map $\alpha \mapsto (\tilde{\eta}_0(\alpha), \eta_1(\alpha))$ (see Theorem 4.4).

In Section 2 we will define $\text{OrderExt}(K_1(A), K_0(A))$ and the homomorphism

$$\tilde{\eta} : \text{Inn}(A) \rightarrow \text{OrderExt}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A))$$

in detail and in Section 3 we will show that

$$\ker \tilde{\eta} = \text{AInn}(A).$$

In Section 4 we will show that $\tilde{\eta}$ is surjective; thus proving the main result.

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2 OrderExt

Let $A$ be a simple unital $C^*$-algebra and let $T_A$ be the set of tracial states of $A$. Let $\alpha \in \overline{\operatorname{Im}}(A)$ and let $M_\alpha$ be the mapping torus of $\alpha$. For a unitary $u \in M_\alpha$ such that $t \mapsto u(t)$ is (piecewise) $C^1$ and for $\tau \in T_A$, we define

$$\tau(u) = \frac{1}{2\pi i} \int_0^1 \tau(\dot{u}(t)u(t)^*)dt.$$ 

In [6] this is denoted by $\tilde{\Delta}_\alpha(u)$. Since $\tau(\dot{u}(t)u(t)^*) = -\tau(u(t)\dot{u}(t)^*)$, it follows that $\tau(u) \in \mathbb{R}$. If $u, v \in M_\alpha$ are $C^1$-unitaries, we obtain that

$$\tau(uv) = \tau(u) + \tau(v).$$

If $h = h^* \in M_\alpha$ is $C^1$, then we have for $u = e^{2\pi it}h$

$$\tau(u) = \int_0^1 \tau(\dot{h}(t))dt = \tau(h(1)) - \tau(h(0)) = 0,$$

where we have used that $\tau \circ \alpha = \tau$, which follows since $\alpha \in \overline{\operatorname{Im}}(A)$. Thus it follows that $\tau(u)$ is constant on each connected component of the $C^1$-unitary group of $M_\alpha$. By taking the matrix algebras over $M_\alpha$ and using the density of $C^1$-unitaries in the unitary group, we obtain a homomorphism $\tau : K_1(M_\alpha) \to \mathbb{R}$ by $[u] \mapsto \tau(u)$ for each $\tau \in T_A$. Since $\tau \in T_A \mapsto \tau(u)$ is affine and continuous, we thus obtain:

Lemma 2.1 For $\alpha \in \overline{\operatorname{Im}}(A)$ there exists a homomorphism

$$R_\alpha : K_1(M_\alpha) \to \mathbb{R}$$

by $R_\alpha([u])(\tau) = \tau(u)$, which will be called the rotation map for $\alpha$.

Since $\alpha_* = \operatorname{id}$ on $K_1(A)$, we have the short exact sequence:

$$0 \to K_0(A) \overset{\iota_*}{\to} K_1(M_\alpha) \overset{\pi_*}{\to} K_1(A) \to 0$$

from the short exact sequence of $C^*$-algebras:

$$0 \to \mathbb{K} \overset{\iota}{\to} M_\alpha \overset{q}{\to} A \to 0.$$ 

If $p$ is a projection in $A$, we have that $\iota_*([p]) = [u]$ where $u \in M_\alpha$ is the unitary defined by

$$u(t) = e^{2\pi it}p + 1 - p.$$ 

Thus we obtain:

Lemma 2.2 For $\alpha \in \overline{\operatorname{Im}}(A)$ the following diagram commutes:

$$\begin{array}{ccc}
K_0(A) & \overset{\iota_*}{\to} & K_1(M_\alpha) \\
D \downarrow & \nearrow & \\
\mathbb{R} & & R_\alpha \\
\end{array}$$

where $D$ is the homomorphism of $K_0(A)$ into $\mathbb{R}$ defined by $D([p])(\tau) = \tau(p)$, which will be called the dimension map for $A$.

Let $G_i = K_i(A)$. If

$$0 \to G_0 \overset{\iota}{\to} E \overset{q}{\to} G_1 \to 0$$

is exact, we denote this short exact sequence by $E$, the same symbol at the middle. Let $R$ be a homomorphism of $E$ into $\mathbb{R}$ such that $R \circ \iota = D$. We consider the set of all pairs $(E, R)$, which we call orderextensions for $(G_1, G_0)$. 

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If \((E', R')\) is another orderextension, we say that \((E, R)\) and \((E', R')\) are isomorphic with each other if there is an isomorphism \(\varphi\) of \(E\) into \(E'\) such that \(R = R' \circ \varphi\) and

\[
\begin{array}{c}
0 \longrightarrow G_0 \overset{\iota}{\longrightarrow} E \overset{q}{\longrightarrow} G_1 \longrightarrow 0 \\
\downarrow \varphi \quad \downarrow \\
0 \longrightarrow G_0 \overset{\iota'}{\longrightarrow} E' \overset{q'}{\longrightarrow} G_1 \longrightarrow 0
\end{array}
\]

is commutative. Note that if \((E, R)\) and \((E', R')\) are isomorphic, \(E\) and \(E'\) are isomorphic as extensions. We define an addition for such pairs by extending that for extensions as follows. If \((E, R)\) and \((E', R')\) are given, define

\[
\begin{align*}
E'' &= \{(x, y) \in E \oplus E' | q(x) = q'(y)\} / \{(\iota(a), -\iota'(a)) | a \in G_0\} \\
\iota'' : G_0 &\longrightarrow E'' \\
a &\longmapsto [(\iota(a), 0)] \\
q'' : E'' &\longrightarrow G_1 \\
[(x, y)] &\longmapsto q(x) \\
R'' : E'' &\longrightarrow \text{Aff}(T_A) \\
[(x, y)] &\longmapsto R(x) + R'(y).
\end{align*}
\]

It is easy to show that these objects are well defined,

\[
\begin{array}{c}
0 \longrightarrow G_0 \overset{\iota''}{\longrightarrow} E'' \overset{q''}{\longrightarrow} G_1 \longrightarrow 0
\end{array}
\]

is exact, and \(R'' \circ \iota'' = D\). The sum of \((E, R)\) and \((E', R')\) is defined to be \((E'', R'')\). Again it is easy to show that the isomorphism classes of those orderextensions form an abelian semigroup. Then the identity element for this semigroup is given by the isomorphism class \([([E_0, R_0])\) of the trivial orderextension \((E_0, R_0)\) given by:

\[
\begin{align*}
E_0 &= G_0 \oplus G_1 \\
\iota_0 : G_0 &\longrightarrow E_0 \\
a &\longmapsto (a, 0) \\
q_0 : E_0 &\longrightarrow G_1 \\
(a, b) &\longmapsto b \\
R_0 : E_0 &\longrightarrow \text{Aff}(T_A) \\
(a, b) &\longmapsto D(a).
\end{align*}
\]

The inverse of \([[(E, R)]\) is given by \([[(E', R')]]\) where

\[
\begin{align*}
E' &= E \\
\iota' &= -\iota \\
q' &= q \\
R' &= -R.
\end{align*}
\]

Thus the semigroup is a group, which we denote by \(\text{OrderExt}(G_1, G_0)\).

Note that \(\text{OrderExt}(G_1, G_0)\) depends also on the dimension map \(D : G_0 \longrightarrow \text{Aff}(T_A)\).

**Lemma 2.3** The map

\[
\tilde{\eta}_0 : \text{Inn}(A) \longrightarrow \text{OrderExt}(K_1(A), K_0(A)) \\
\alpha \longmapsto [([K_1(M_{\alpha}), R_{\alpha}])
\]

is a homomorphism.
Proof. By Lemma 2.2 $\gamma_0$ is well-defined.

Let $\alpha, \beta \in \text{Im}(A)$ and $(E, R)$ be the sum of $(K_1(M_\alpha), R_\alpha)$ and $(K_1(M_\beta), R_\beta)$. We have to show that $(E, R)$ is isomorphic to $(K_1(M_{\alpha\beta}), R_{\alpha\beta})$.

Let $g \in K_1(M_\alpha)$ and $h \in K_1(M_\beta)$ such that $q(g) = q(h)$. Let $v \in M_\alpha \otimes M_\alpha$ and $w \in M_\alpha \otimes M_\beta$ be unitaries such that $[v] = g$, $[w] = h$, and $v(0) = w(0)$. Then we define a unitary $u \in M_\alpha \otimes M_{\alpha\beta}$ by

$$u(t) = \begin{cases} v(2t) & \text{if } 0 \leq t \leq 1/2 \\ \alpha(w(2t - 1)) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Then $[u] \in K_1(M_{\alpha\beta})$ depends only on $[v]$ and $[w]$. Thus we have a map $\varphi$ of

$$\{(g, h) \in K_1(M_\alpha) \oplus K_1(M_\beta) \mid q(g) = q(h)\}$$

to $K_1(M_{\alpha\beta})$. It is easy to show that $\varphi$ is a surjective homomorphism and the kernel of $\varphi$ equals $\{(\iota(a), -\iota(a)) \mid a \in K_0(A)\}$. Hence $\varphi$ induces an isomorphism $\phi : E \rightarrow K_1(M_{\alpha\beta})$. Since

$$R_{\alpha\beta}([u]) = R_\alpha([v]) + R_\beta([w])$$

for the above $u$, $(E, R)$ is isomorphic to $(K_1(M_{\alpha\beta}), R_{\alpha\beta})$.

Lemma 2.4 If $(E, R)$ is an orderextension for $(G_1, G_0)$ and $\text{Range } R = \text{Range } D$, then

$$0 \rightarrow \ker D \xrightarrow{\iota_* |_{\ker D}} \ker R \xrightarrow{q_* |_{\ker R}} G_1 \rightarrow 0$$

is exact.

Proof. It is obvious that the above sequence is well-defined, the compositions of two consecutive maps vanish, and it is exact at $\ker D$. Let $g \in \ker R$ with $q_*(g) = 0$. Then there is a $g' \in G_0$ such that $\iota_*(g') = g$. But, since $D(g') = R(g) = 0$, we have that $g' \in \ker D$, which implies that it is exact at $\ker R$. Let $g \in G_1$. Then there is a $g' \in E$ with $q_*(g') = g$ and there must be a $g'' \in G_0$ such that $D(g'') = R(g')$. Since $q_*(g' - \iota_*(g'')) = g$ and $R(g' - \iota_*(g'')) = 0$, we have that $g \in \text{Range}(q_* |_{\ker R})$.

Proposition 2.5 If $(E, R)$ is an orderextension for $(G_1, G_0)$, the following conditions are equivalent:

i. $[(E, R)] = 0$,

ii. (a) $0 \rightarrow G_0 \rightarrow E \rightarrow G_1 \rightarrow 0$ is trivial,

(b) $\text{Range } R = \text{Range } D$,

(c) $0 \rightarrow \ker D \rightarrow \ker R \rightarrow G_1 \rightarrow 0$ is trivial,

iii. $0 \rightarrow \ker D \rightarrow \ker R \rightarrow G_1 \rightarrow 0$ is exact and trivial.

Proof. If $(E_0, R_0)$ is the trivial orderextension, it satisfies (2). Any orderextension isomorphic to $(E_0, R_0)$ also satisfies (2). Thus (1) implies (2).

Suppose that $(E, R)$ satisfies (2). Note that the sequence in (c) is exact by 2.4. By (c) there is a homomorphism $\nu$ of $G_1$ into $\ker R$ such that $q \circ \nu = \id$. Hence $E = \iota(G_0) \oplus \nu(G_1)$ and $R$ is given by

$$\iota(G_0) \oplus \nu(G_1) \rightarrow \text{Aff}(T_A)$$

$$a + \nu(b) \mapsto D(a).$$

Thus $(E, R)$ is isomorphic to the trivial orderextension, i.e., (2) implies (1).

It follows from 2.4 that (2) implies (3). The converse also follows from the arguments in the previous paragraph.

Remark 2.6 By the Thom isomorphism [5], $K_1(M_\alpha)$ is isomorphic to $K_{\iota+1}(A \times_\alpha \mathbb{Z})$ as an abelian group. By extending $\tau \in T_A$ to a tracial state of $A \times_\alpha \mathbb{Z}$ and defining a natural map $D_{\alpha} : K_0(A \times_\alpha \mathbb{Z}) \rightarrow \text{Aff}(T_A)$, it follows that $(K_1(M_\alpha), R_\alpha)$ is isomorphic to $(K_0(A \times_\alpha \mathbb{Z}), D_\alpha)$ [5]. See also [6, 12, 1].
3 Asymptotically inner automorphisms

From now on we will assume that the \( C^* \)-algebra \( A \) is a simple unital AT algebra of real rank zero. In this case by Elliott’s result [7] \( A \) is determined by \((K_0(A), [1], K_1(A)) \) up to isomorphism, where \( K_0(A) \) is a dimension group, \( K_1(A) \) is a torsion-free abelian group, and \([1] \in K_0(A)^+\). Note that the tracial state space \( T_A \) of \( A \) is identified with the compact convex set of order-preserving homomorphisms \( f : K_0(A) \to \mathbb{R} \) with \( f([1]) = 1 \).

Let \( \alpha \in \overline{\text{Inn}}(A) \). We recall that \( \alpha \) is asymptotically inner if there exists a continuous map \( v : [0, 1] \to U(A) \) such that

\[
\alpha(a) = \lim_{t \to 1} \text{Ad} \, v_t(a), \quad a \in A.
\]

We denote by \( \overline{\text{AInn}}(A) \) the group of asymptotically inner automorphisms of \( A \). We also recall that \( \tilde{\eta} \) is the homomorphism of \( \overline{\text{Inn}}(A) \) into

\[
\text{OrderExt}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A))
\]

defined by \( \alpha \mapsto \tilde{\eta}_0(\alpha) \oplus \eta_1(\alpha) \).

Before stating the main theorem of this section, let us recall the notion of Bott element for pairs of almost commuting unitaries in a unital \( C^* \)-algebra \( A \) [10, 11]: Given \( u, v \in U(A) \) with \( [u, v] \equiv uv - vu \approx 0 \), we associate \( B(u, v) \in K_0(A) \), which is the equivalence class of a projection close to the image of the Bott projection in \( M_2 \otimes C(T^2) \) under the quasi-homomorphism from \( M_2 \otimes C(T^2) \) into \( M_2 \otimes A \) mapping the two canonical unitaries of \( C(T^2) \) into \( u, v \) respectively. If \( A = M_n \), this can also be given by

\[
B(u, v) = \frac{1}{2\pi i} \text{Tr}(\log uv^*u^*) \in \mathbb{Z} = K_0(M_n),
\]

where \( \log \) is the logarithm with values in \( \{ z : \text{Im}(z) \in (-\pi, \pi) \} \). (That \( B(u, v) \) is an integer follows from the fact that the determinant of \( uv^*u^* \) is 1.) We note that \( B(u, v) \) is invariant under homotopy of pairs of almost commuting unitaries and that \( B(u, v) = -B(u^*, v) = -B(v, u), B(u, v_1v_2) = B(u, v_1) + B(u, v_2) \). We quote [4] for another characterization of the Bott element, which is used to prove the following result we will need later: If \( A \) is a simple unital AT algebra of real rank zero and \( u, v \in U(A) \) satisfy that \([u, v] \approx 0, B(u, v) = 0, \text{Sp}(v) \) is almost dense in \( T \), and \([u] = 0 \), then there is a path \( u_t, \ t \in [0, 1] \) in \( U(A) \) such that \([u_t, v] \approx 0, \ u_0 = 1, \) and \( u_1 = u \).

**Theorem 3.1** Let \( A \) be a simple unital AT algebra of real rank zero and let \( \alpha \in \overline{\text{Inn}}(A) \). Then the following conditions are equivalent:

i. \( \tilde{\eta}(\alpha) = 0 \),

ii. \( \alpha \in \overline{\text{AInn}}(A) \).

**Proof of (2) ⇒ (1)**

Since \( \eta \) is homotopy invariant, \( \eta(\alpha) = (\eta_0(\alpha), \eta_1(\alpha)) = 0 \) in \( \text{Ext}(K_1(A), K_0(A)) \oplus \text{Ext}(K_0(A), K_1(A)) \).

We may suppose that we have a piecewise \( C^1 \) map \( v \) of \([0, 1] \) into \( U(A) \) such that \([u_t, v] \approx 0, \ u_0 = 1, \) and \( u_1 = u \).

Let \( u \in U(A) \). We define a unitary \( \tilde{u} \in M_2 \otimes M_2 \) by composing the following paths:

\[
[0, 1] \ni t \mapsto R_t \left( \begin{array}{cc} 1 & 0 \\ 0 & v_0 \end{array} \right) R_t^{-1} \left( \begin{array}{cc} u & 0 \\ 0 & 1 \end{array} \right) R_t \left( \begin{array}{cc} 1 & 0 \\ 0 & v_0^* \end{array} \right) R_t^{-1}
\]

and

\[
[0, 1] \ni t \mapsto \left( \begin{array}{cc} v_t u v_t^* & 0 \\ 0 & 1 \end{array} \right)
\]

with

\[
1 \mapsto \left( \begin{array}{cc} \alpha(u) & 0 \\ 0 & 1 \end{array} \right),
\]
where
\[ R_t = \begin{pmatrix} \cos \frac{\pi}{2} t & -\sin \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t & \cos \frac{\pi}{2} t \end{pmatrix}. \]

Then it follows that \( \tau(\hat{u}(t)\hat{u}(t)^*) = 0 \) for \( t \in T_A \). In particular \( R_\alpha([\hat{u}]) = 0 \). Since \( q_\alpha([\hat{u}]) = [u] \), the map \([u] \mapsto [\hat{u}]\) defines a homomorphism \( \varphi \) of \( K_1(A) \) into \( \ker R_\alpha \) such that \( q_\alpha \circ \varphi = \text{id} \). This implies that
\[ 0 \longrightarrow \ker D \longrightarrow \ker R_\alpha \longrightarrow K_1(A) \longrightarrow 0 \]
is exact and trivial, and thus concludes the proof by 2.5.

The rest of this section will be devoted to the proof of (1) \( \Rightarrow \) (2).

Let \( \{A_n\} \) be an increasing sequence of \( T \) subalgebras of \( A \) such that \( A_1 \ni 1 \) and \( A = \bigcup_{n=1}^{\infty} A_n \). We express \( A_n \) as
\[ A_n = \bigoplus_{i=1}^{k_n} B_{n,i} \otimes C(T) \]
where \( B_{n,i} \) is isomorphic to the full matrix algebra \( M_{[n,i]} \). By identifying \( K_1(A) \) with \( \mathbb{Z}^{k_n} \) in a natural way we obtain a homomorphism \( K_1(A_n) \) into \( K_1(A_{n+1}) \) as the multiplication of a matrix \( \chi_n \). We always assume that \( \chi_n^0(i,j) \) is big and \( |\chi_n^0(i,j)|/\chi_n^1(i,j) \) is small compared with 1 and that the embedding of \( A_n \) into \( A_{n+1} \) is in standard form, i.e., \( B_n = \bigoplus_{i=1}^{k_n} B_{n,i} \subset B_{n+1} \) and the canonical unitary \( z_n \) of \( 1 \otimes C(T) \subset A_n \) in \( B_{n+1} \cap B_n \subset C(T) \) is a direct sum of elements of the form:
\[ \begin{pmatrix} 0 & z_{n+1} \\ 1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 0 \end{pmatrix} \]
with \( L = \pm 1 \); e.g., if \( \chi_n^1(i,j) > 0 \), \( z_{n+1} = z_{n+1} \) is a direct sum of \( \chi_n^1(i,j) \) matrices of the above form with \( L = 1 \)
in \( B_{n+1} \cap B_n \). Let \( \chi_n^0(i,j) = M_{\chi_n^0(i,j)} \otimes C(T) \) [7, 11]. For each \( n = 1, 2, \ldots \) let
\[ M_{n,n} = \{x \in C[0,1] \otimes A | x(0) \in A_n, \alpha(x(0)) = x(1)\}. \]
Then we obtain the exact sequence of \( C^* \)-algebras:
\[ 0 \longrightarrow (\text{SA}) \xrightarrow{i_n} M_{n,n} \xrightarrow{q_n} A_n \longrightarrow 0 \]
from which follow the exact sequences of abelian groups:
\[ 0 \longrightarrow K_1(A) \longrightarrow K_{n+1}(M_{n,n}) \longrightarrow K_{n+1}(A_n) \longrightarrow 0. \]

Since \( K_1(A_n) \cong \mathbb{Z}^{k_n} \), the above extensions are all trivial.

Let \( R = R_{\alpha} \) and \( R_n = R \circ j_{n*} : K_1(M_{n,n}) \to \text{Aff}(T_A) \), where \( j_n \) is the embedding of \( M_{n,n} \) into \( M_n \). Since Range \( D \) = Range \( R_n \), we obtain by 2.4 that
\[ 0 \longrightarrow \ker D \xrightarrow{i_n} \ker R_n \xrightarrow{q_n*} K_1(A_n) \longrightarrow 0 \]
is exact. Note that the inductive limit of these extensions is naturally isomorphic to the exact sequence:
\[ 0 \longrightarrow \ker D \longrightarrow \ker R \longrightarrow K_1(A) \longrightarrow 0. \]
We shall specify a homomorphism \( \varphi_n \) of \( K_1(A_n) \) into \( \ker R_n \) such that
\[ q_{n*} \circ \varphi_n = \text{id}. \]
Since $\alpha \in \mathrm{Inn}(A)$, we have a $u_n \in U(A)$ for each $n$ such that
\[
\alpha | B_n = \mathrm{Ad} u_n | B_n, \\
\alpha(z_n) \approx \mathrm{Ad} u_n(z_n),
\]
where $B_n = \bigoplus_{i=1}^{k_n} B_{ni} \subset A_n$ and $z_n$ is the canonical unitary of $C(T) \subset A_n$. Define
\[
h_{ni} = \frac{1}{2\pi i} \log \alpha(z_n) \mathrm{Ad} u_n(z_n)^*
\]
where $z_{ni} = z_n p_{ni} + 1 - p_{ni}$ with $p_{ni}$ the identity of $B_{ni}$ and $h_{ni} = h_{ni}^*$ is defined uniquely as $\|h_{ni}\| \approx 0$ since $\alpha(z_n) \mathrm{Ad} u_n(z_n) \approx 1$. Define $\zeta_{ni} \in U(M_{\alpha,n} \otimes M_2)$ by composing two paths of unitaries:
\[
[0, 1] \ni t \mapsto R_t(u_n^* \oplus 1) R_t^{-1}(u_n \oplus 1)(u_n^* \oplus 1) R_t(u_n \oplus 1) R_t^{-1}
\]
and
\[
[0, 1] \ni t \mapsto e^{2\pi i t h_{ni}} \mathrm{Ad} u_n(z_n) \oplus 1.
\]
Then we have that
\[
q_n(\zeta_{ni}) = z_n \oplus 1, \\
R_n(\zeta_{ni}) = \hat{h}_{ni},
\]
where $\hat{h}_{ni} \in \mathrm{Aff}(T_A)$ is defined by
\[
\hat{h}_{ni}(\tau) = \tau(h_{ni}), \quad \tau \in T_A.
\]
Since the above procedure applies to a unitary $z_n p + 1 - p$ with $p$ a minimal projection in $B_{ni}$, it follows that $[\zeta_{ni}] \in K_1(M_{\alpha,n})$ is divisible by $[n, i]$. Thus one obtains a homomorphism $\varphi_n$ of $K_1(A_n)$ into $K_1(M_{\alpha,n})$ with $q_n \circ \varphi = \mathrm{id}$ by setting
\[
\varphi_n : [z_{ni}] \mapsto [\zeta_{ni}].
\]

**Lemma 3.2** Range $D$ is dense in $\mathrm{Aff}(T_A)$.

**Proof.** Since $A$ is a simple unital AT algebra of real rank zero, it is approximately divisible [8]. Thus this is 3.14(a) of [3]. (A unital $C^*$-algebra is approximately divisible if it has a central sequence $\{B_n\}$ of unital $C^*$-subalgebras with $B_n \cong M_2 \oplus M_3$ [3]. Since $A$ is obtained as the inductive limit of $\{A_n\}$ of $T$ algebras with unital embeddings and the embeddings need to satisfy only the K-theoretic conditions and the condition of real rank zero [2], thanks to Elliott’s result [7], we can easily arrange the inductive system so that $A_{n+1} \cap A'_n \supset M_2 \oplus M_3$, which implies that $A$ is approximately divisible.)

Let
\[
\delta_n = \min \{\tau(p_{ni}) \mid \tau \in T_A\},
\]
where $p_{ni}$ is the identity of $B_{ni}$. Since $A$ is simple, $\delta_n$ is strictly positive. We choose the unitary $u_n \in A$ so that $\|h_{ni}\| < \delta_n$. Since Range $R_n = \text{Range } D$, we have, for any $\epsilon > 0$ with $\|h_{ni}\| + \epsilon < \delta_n$, projections $p_{\pm} \in A$ such that
\[
\frac{1}{[n, i]} h_{ni} = D(p_+) - D(p_-), \\
\|D(p_{\pm})\| < \frac{1}{[n, i]} (\|h_{ni}\| + \epsilon),
\]
where $D$ is also regarded as a map of the projections into $\mathrm{Aff}(T_A)$. (First we approximate $h_{ni+}/[n, i]$ by $D(p_+)$ with $p_+$ a projection such that $D(p_+) - h_{ni+}/[n, i] > 0$ (strictly positive), where $h_{ni+}$ is the positive part of $h_{ni}$. We should note that $\|h_{ni+}/[n, i]\| \leq \|h_{ni}\|/[n, i]$ and find a projection $p_-$ such that $D(p_-) = D(p_+) - h_{ni}/[n, i] \approx h_{ni-}/[n, i]$.) Since $D(p_{\pm}) < \delta_n/[n, i] \leq D(p_{\pm})/[n, i]$, we find projections $e_{i, \pm} \in p_{ni} A p_{ni} \cap B_{ni}'$ such that
\[
\hat{h}_{ni} = D(e_{i, +}) - D(e_{i, -}), \\
\|D(e_{i, \pm})\| < \|h_{ni}\| + \epsilon.
\]
Thus, by making \( \|h_{n1}\| \) small, we can make \( \|D(e_{1,\pm})\| \) arbitrarily small. Then, by using Lemma 3.4 below, we can find a unitary \( w_{n1} \in p_{n1}Ap_{n1} \cap B_{n1}' \) such that

\[
\begin{align*}
w_{n1} &= w_{n1}p_{n1} + 1 - p_{n1}, \\
\text{Ad} w_{n1}(z_{n1}) &\approx z_{n1}, \quad \text{(in the order of } \|h_{n1}\|) \\
\hat{k}_{n1} &= h_{n1},
\end{align*}
\]

where

\[
\hat{k}_{n1} = \frac{1}{2\pi i} \log \text{Ad} w_{n1}(z_{n1}) z_{n1}^*.
\]

Let \( w_n = w_{n1}w_{n2} \cdots w_{nk_n} \). Note that

\[
\alpha(z_{n1}) \text{Ad} u_n w_n(z_{n1}^*) = \alpha(z_{n1}) \text{Ad} u_n(z_{n1}^*) \text{Ad} u_n(z_{n1} \text{Ad} w_n(z_{n1}^*)) = e^{2\pi i k_{n1}} \text{Ad} u_n(e^{-2\pi i k_{n1}}).
\]

Then composing the two paths:

\[
[0, 1] \ni t \mapsto \text{Ad} u_n(e^{-2\pi i k_{n1}})
\]

and

\[
[0, 1] \ni t \mapsto e^{2\pi i h_{n1}} \text{Ad} u_n(e^{-2\pi i k_{n1}})
\]

multiplied with \( \text{Ad} u_n w_n(z_{n1}) \) to the right, we obtain a path \( U \) from \( \text{Ad} u_n w_n(z_{n1}) \) to \( \alpha(z_{n1}) \) such that

\[
\frac{1}{2\pi i} \int_0^1 \tau(\dot{U}(t)U(t)^*) dt = 0, \quad \tau \in T_A.
\]

Since \( U \) is in a small neighbourhood of \( \alpha(z_{n1}) \cong \text{Ad} u_n w_n(z_{n1}) \), it follows that the unitary \( \zeta_{n1} \) obtained from \( z_{n1} \) in the same way as before with \( u_n w_n \) in place of \( u_n \) satisfies

\[
R_n([\zeta_{n1}]) = 0.
\]

Thus we have shown:

**Lemma 3.3** Suppose that \( \tilde{\eta}_0(\alpha) = 0 \). Then for any \( n \) and \( \epsilon \in (0, 1) \) there exists a unitary \( u_n \in A \) such that

\[
\begin{align*}
\alpha|B_n &= \text{Ad} u_n|B_n, \\
\|\alpha(z_{n1}) - z_{n1}\| &< \epsilon, \\
\hat{h}_{n1} &= 0,
\end{align*}
\]

where

\[
h_{n1} = \frac{1}{2\pi i} \log \alpha(z_{n1}) \text{Ad} u_n(z_{n1}^*).
\]

Hence defining a unitary \( \zeta_{n1} \in M_{\alpha,n} \otimes M_2 \) by composing the two paths:

\[
[0, 1] \ni t \mapsto R_t(u_n^* \oplus 1) R_t^{-1} \text{Ad} u_n(z_{n1}) \oplus 1 R_t(u_n \oplus 1) R_t^{-1}
\]

and

\[
[0, 1] \ni t \mapsto e^{2\pi i h_{n1}} \text{Ad} u_n(z_{n1}) \oplus 1,
\]

where \( R_t \) is defined as before, one can define a homomorphism \( \varphi_n \) of \( K_1(A_n) \) into \( \ker R_n \) by \( \varphi([z_{n1}]) = [\zeta_{n1}], \quad i = 1, \ldots, k_n. \)

**Lemma 3.4** If \( e \in p_{n1}Ap_{n1} \cap B_{n1}' \) is a projection such that \( \|D(e)\| \) is sufficiently small, then for any \( \epsilon > 0 \) there exists a unitary \( w_{\pm} \in p_{n1}Ap_{n1} \cap B_{n1}' \) such that

\[
\begin{align*}
\|\text{Ad} w_{\pm}(z_{n1}) - z_{n1}\| &< 2\pi \|D(e)\| + \epsilon, \\
|w_{\pm}| &= 0, \\
B(w_{\pm}, z_{n1}) &= \pm \epsilon.
\end{align*}
\]

In particular if \( k_{\pm} = \frac{1}{2\pi} \log \text{Ad} z_{n1} w_{\pm}(z_{n1}^*) \), it follows that \( \hat{k}_{\pm} = \pm D(e) \).
Proof. To simplify the notation we may suppose that \( p_m A_m p_m \cap B_m \) to be \( A \) and \( z_m \) to be the canonical unitary \( z_1 \in A_1 = C(T) \).

Since the projection \( e \) plays a role only through \([e]\), we may suppose that \( e \in A_m \) for some \( m > 1 \). We will later assume that \( m \) is sufficiently large. Since \( A_n \hookrightarrow A_{n+1} \) are in the standard form, \( z_1 p_m \) in \( B_{m} \otimes C(T) \) looks like a direct sum of elements of the form:

\[
\begin{pmatrix}
0 & L_s \\
1 & & \ddots & \ddots \\
& & 1 & 0
\end{pmatrix} \in M_{M_s}(C(T))
\]

where \( L_s = \pm 1, \ M_s \gg 1 \) and

\[
\sum_s L_s = \chi_{m_1}^1(j, 1), \quad \sum_s M_s = \chi_{m_1}^0(j, 1) = [m, j].
\]

Note that \( D(e) \) takes values in the convex hull of

\[
\frac{\dim(\epsilon p_{mj})}{[m, j]}, \ j = 1, \ldots, k_m,
\]

which are all assumed to be much less than 1. Let \( t_m \) be the maximum of these \( k_m \) values. Then \( t_m \) decreases as \( m \to \infty \) and the limit of \( t_m \) equals \( \tau(e) \) for some \( \tau \in T_A \) (or \( \|D(e)\| \)). Thus if \( m \) is sufficiently large, we may assume that \( t_m < \|D(e)\| + \epsilon/4\pi \). We can obtain the required unitary \( w_j \) in \( B_{m} \otimes C(T) \) as the direct sum of elements of the form:

\[
\begin{pmatrix}
1 & \omega & & \\
& \omega^2 & & \\
& & \ddots & \\
& & & \omega^{M_s-1}
\end{pmatrix}
\]

where \( \omega = e^{-2\pi iN_s/M_s} \) and the integers \( N_s \) are chosen so that

\[
\sum N_s = \dim(\epsilon p_{mj}), \quad \frac{N_s}{M_s} = \frac{\dim(\epsilon p_{mj})}{[m, j]}.
\]

Note that by defining

\[
k_j = \frac{1}{2\pi i} \log z_1 p_{mj} A w_j (z_1^* p_{mj}),
\]

the Bott element \( B(w_j, z_1 p_{mj}) \in K_0(A_{mj} p_m) = \mathbb{Z} \) for the almost commuting pair \( w_j, z_1 p_{mj} \) of unitaries in \( A_{mj} p_m = B_{mj} \otimes C(T) \) is equal to

\[
\text{Tr}(k_j) = \text{Tr}(\oplus_s \frac{N_s}{M_s} 1_s) = \sum N_s = \dim(\epsilon p_{mj}),
\]

where \( k_j \in B_{mj} \otimes C(T) \) should be evaluated at some (or any) point of \( T \) (see [10, 11, 4]). This shows that

\[
B(w_j, z_1 p_{mj}) = [\epsilon p_{mj}],
\]

and in particular that \( \hat{k}_j = D(\epsilon p_{mj}) \).

If \( m \) is sufficiently large or all \( M_s \) are sufficiently large, we can assume that

\[
\frac{N_s}{M_s} < \|D(e)\| + \epsilon/2\pi.
\]
Thus we obtain the norm estimate
\[ \| \text{Ad} w_j(z_1 p_m) - z_1 p_m \| < 2\pi \| D(e) \| + \epsilon. \]
By taking \( w_+ = w_1 + w_2 + \cdots + w_{k_n} \), this completes the proof for \( w_+ \). For \( w_- \) we just replace \( \omega \) in the definition of \( w_j \) by \( z = e^{2\pi i n_j/M_n} \).

By defining \( \varphi_n : K_1(A_n) \to \ker R_n \) as above, we identify \( \ker R_n \) with \( \ker D \oplus K_1(A_n) \). We now have to translate the natural map \( \ker R_n \to \ker R_{n+1} \) into the map \( \psi_n : \ker D \oplus K_1(A_n) \to \ker D \oplus K_1(A_{n+1}) \):
\[
\begin{align*}
0 & \to \ker D \oplus K_1(A_n) \to 0 \\
0 & \to \ker D \oplus K_1(A_{n+1}) \to 0
\end{align*}
\]
where we have used that \( \psi_n \) must be of the form \( \psi_n(a, b) = (a + \psi_n^0(b), \chi_n^1(b)) \).

**Lemma 3.5** If \( u_n \) is a unitary in \( A \) and \( \epsilon \in (0, 1) \) such that
\[
\alpha|B_n = \text{Ad} u_n|B_n, \\
\| \alpha(z_n) - \text{Ad} u_n(z_n) \| < \epsilon, \\
h_{n,i} = 0,
\]
then for any \( m \leq n \) and \( j = 1, \ldots, k_m \),
\[
\| \alpha(z_m) - \text{Ad} u_n(z_m) \| < \epsilon, \quad (\ast)
\]
\[
h_{m,j} = 0, \quad (\ast\ast)
\]
where
\[
h_{m,j} = \frac{1}{2\pi i} \log \alpha(z_m) \text{Ad} u_n(z_m).
\]

**Proof.** By the assumption on the embedding of \( A_n \) into \( A_0 \), \( (\ast) \) follows immediately. Since the homomorphism \( \varphi_n : K_1(A_n) \to \ker R_n \) can be defined on \([z_m]\) in the canonical way and \( R_n \varphi_n([z_m]) = \tilde{h}_m \), \( (\ast\ast) \) also follows immediately.

**Lemma 3.6** The homomorphism \( \psi_n^0 : K_1(A_n) \to \ker D \) is given by
\[
[z_n] \mapsto B(u_{n+1}^* u_n, z_n),
\]
where \([z_n] = [u, i] e_i \) with \( (e_i) \), the canonical basis for \( Z^n = K_1(A_n) \) and \( B(u_{n+1}^* u_n, z_n) \) is divisible by \([u, i]\).

**Proof.** First of all we shall show that \( D(B(u_{n+1}^* u_n, z_n)) = 0 \). Because if we define self-adjoint \( h \in A \) by
\[
\begin{align*}
h_1 &= \frac{1}{2\pi i} \log \alpha(z_n) \text{Ad} u_{n+1}^* u_n(z_n), \\
h_2 &= \frac{1}{2\pi i} \log \alpha(z_n) \text{Ad} u_n(z_n), \\
h_3 &= \frac{1}{2\pi i} \log z_n \text{Ad} (u_{n+1}^* u_n)(z_n)
\end{align*}
\]
then \( \hat{h}_2 = 0 \) and \( \hat{h}_1 = 0 \) by 3.5 and hence \( \hat{h}_3 = 0 \) since
\[
\text{Ad} u_{n+1} (e^{2\pi i h_3}) = e^{-2\pi i h_1} e^{2\pi i h_2}.
\]
One way of proving that \( \hat{h}_3 = 0 \) is to take a closed path \( w \) of unitaries:
\[
w(t) = \begin{cases} 
  e^{-6\pi i h_1} & 0 \leq t \leq 1/3 \\
  e^{-2\pi i h_1} e^{2\pi i (3t-1) h_2} & 1/3 \leq t \leq 2/3 \\
  e^{-2\pi i h_1} e^{2\pi i h_2} \text{Ad} u_{n+1} (e^{2\pi i (3t-2) h_3}) & 2/3 \leq t \leq 1
\end{cases}
\]

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in a neighbourhood of 1, and compute for any $\tau \in T_A$, $0 = 1/2\pi i \int_{-1}^{1} \tau(w(t)w(t)) dt = -\tau(h_1) + \tau(h_2) - \tau(h_3).$

We may suppose that $u_{n+1}^* u_n \in A_m \cap B'_n$ for some $m > n$. In this case $B(u_{n+1}^* u_n, z_{ni})$ in $K_0(A_m)$ is defined by

$$(\text{Tr}_{B_{m_j}}(h_{3p_{m_j}}))_j,$$

where $h_{3p_{m_j}} \in B_{m_j} \otimes C(\mathbf{T})$ is evaluated at a point of $\mathbf{T}$ and $\hat{h}_3 = 0$ means that for any $\tau \in T_A$,

$$\sum_j \tau(p_{m_j}) \frac{\text{Tr}_{B_{m_j}}(h_{3p_{m_j}})}{[m, j]} = 0.$$

Define a path $v_{nt}$, $t \in [0, 1]$ of unitaries in $A \otimes M_2$ by

$$v_{nt} = R_t(u_n^* \otimes 1)R_t^{-1}(u_n \otimes 1).$$

Then to compute $\psi_n^0([z_{ni}])$ we have to calculate

$$\psi_n^0([z_{ni}]) = \varphi_n([z_{ni}]) - \varphi_{n+1}([z_{ni}]) = [t \mapsto \text{Ad} v_{nt}(z_{ni})] - [t \mapsto \text{Ad} v_{nt+1}(z_{ni})] \quad (*)$$

in $K_1(M_{a,n+1})$ where $z_{ni}$ is identified with $z_{ni} \otimes 1$ (see 2.8 of [11] for a similar computation). More precisely we have to add a short path from $\text{Ad} u_n(z_{ni})$ (resp. $\text{Ad} u_{n+1}(z_{ni})$) to $\alpha(z_{ni})$ to the path $t \mapsto \text{Ad} v_{nt}(z_{ni})$ (resp. $t \mapsto \text{Ad} v_{nt+1}(z_{ni})$) to get a unitary in $M_{a,n+1} \otimes M_2$ and we always understand the formulae in this way. Note that $(*)$ is equal to

$$[t \mapsto \text{Ad} v_{nt}(z_{ni})\text{Ad} v_{nt+1}(z_{ni}^*)]$$

in $K_1(SA) \subset K_1(M_{a,n+1})$ or, by applying $t \mapsto \text{Ad} v_{nt+1,t}$, which induces the identity map on $K_1(SA)$, to

$$[t \mapsto v_{nt+1,t}^* v_{nt} v_{nt+1,t} v_{nt}^*].$$

Since

$$v_{nt}^{*} = (u_{n+1}^* \otimes 1)R_t(u_{n+1} u_n^* \otimes 1)R_t^{-1}(u_n \otimes 1),$$

the above element is equal to the class of

$$t \mapsto (u_{n+1}^* z_{ni} u_n^* \otimes 1)R_t(u_{n+1}^* u_n^* \otimes 1)R_t^{-1}(u_n^* z_{ni} u_n \otimes 1)R_t(u_n^* u_{n+1} \otimes 1)R_t^{-1}$$

by applying $\text{Ad}(u_{n+1}^* z_{ni}^* \otimes 1)$. Again this is equal to the class of

$$t \mapsto u_{n+1}^* u_n^* \otimes 1)R_t(u_{n+1}^* u_{n+1} \otimes 1)R_t^{-1}(z_{ni} \otimes 1)R_t(u_{n+1}^* u_{n+1} \otimes 1)R_t^{-1}$$

by applying $t \mapsto \text{Ad}(u_{n+1}^* \otimes 1)$. More precisely we have to add a short path to connect the value at $t = 1$ to $1$. Since $u_{n+1}^* u_n \in A_m \cap B'_n$ by the assumption, the path can be taken in $A_m$. The above element in $K_1(SA_m) = K_0(A_m)$ is equal to

$$\left(-\frac{1}{2\pi i} \text{Tr}_{B_{m_j}} \log(u_{n+1}^* z_{ni} u_n^* u_{n+1}^* z_{ni} p_{m_j})\right)_j$$

$$= \left(\frac{1}{2\pi i} \text{Tr}_{B_{m_j}} \log(z_{ni} u_{n+1}^* u_{n+1}^* u_{n+1}^* z_{ni}^* p_{m_j})\right)_j$$

$$= B_{Am}(u_{n+1}^* u_n, z_{ni}).$$

Note also that since the non-trivial part of $z_{ni} u_{n+1}^* u_{n+1}^* z_{ni}^*(u_{n+1}^* u_n)$ belongs to $p_{ni}A_{mp_{ni}} \cap B'_{ni}$, each component of $B_{Am}(u_{n+1}^* u_n, z_{ni})$ is divisible by $[n, i]$. Then we obtain that

$$\psi_n^0([z_{ni}]) = B(u_{n+1}^* u_n, z_{ni}), \quad i = 1, \ldots, k_n,$$

is a well-defined homomorphism of $K_1(A_n)$ into $\ker D \subset K_0(A)$. 12
Lemma 3.7 Suppose that \( \overline{q}_0(\alpha) = 0 \). Then there exist unitaries \( u_n \in A \) such that
\[
\alpha|B_n = \text{Ad } u_n|B_n, \\
||\alpha(z_m) - \text{Ad } u_n(z_m)|| < 2^{-n}, \ m \leq n, \\
B(u_{n+1}^* u_n, z_n) = 0, \ i = 1, \ldots, k_n, \\
\hat{h}_{ni} = 0, \ i = 1, \ldots, k_n,
\]
where
\[
h_{ni} = \frac{1}{2\pi i} \log \alpha(z_{ni}) \text{Ad } u_n(z_{ni}).
\]

Proof. By the assumption and Proposition 2.5 the sequence of trivial extensions:
\[
0 \rightarrow \ker D \rightarrow \ker D \oplus K_1(A_0) \xrightarrow{q_0} K_1(A_0) \rightarrow 0 \\
0 \rightarrow \ker D \rightarrow \ker D \oplus K_1(A_{n+1}) \xrightarrow{q_{n+1}} K_1(A_{n+1}) \rightarrow 0
\]
defines the trivial extension in \( \text{Ext}(K_1(A), \ker D) \). Hence we have a homomorphism \( h_0^0 : K_1(A_0) \rightarrow \ker D \) for each \( n \) such that
\[
\psi_n^0 = h_n^0 - h_{n+1}^0 \chi_n^1.
\]
(To see this we denote by \( E \) the inductive limit of the middle terms, and by \( \varphi \) a homomorphism of \( K_1(A) \) into \( E \) such that \( q_\varphi = \text{id} \). If \( \xi_n \) denotes the natural homomorphism of \( K_1(A_n) \) into \( \ker D \oplus K_1(A_n) \) composed with \( \ker D \otimes K_1(A_n) \rightarrow E \), \( \psi_n^0 \) is given by \( \psi_n^0 = \xi_n - \xi_n + \chi_n^1 \). We set \( h_n^0 = \xi_n - \varphi_n \) where \( \varphi_n \) is the homomorphism \( K_1(A_n) \rightarrow K_1(A) \) composed with \( \varphi : K_1(A) \rightarrow E \). Then it follows that
\[
h_n^0 - h_{n+1}^0 \chi_n^1 = \xi_n - \varphi_n - \xi_n + \chi_n^1 = \xi_n - \xi_n + \chi_n^1 = \psi_n^0,
\]
where we have used that \( \varphi_n = \varphi_n + \chi_n^1 \).

Since \( h_n^0(e_{i\pm}^n) \in \ker D \), where \( (e_{i\pm}^n)_{i=1}^n \) is the canonical basis for \( Z^{k_n} = K_1(A_n) \), we can find projections \( e_{i\pm}^n \in p_{ni}A_{ni} \cap B_{ni}^\prime \) such that
\[
[n, i]h_n^0(e_{i}^n) = [e_{i+}^n] - [e_{i-}^n]
\]
and \( ||D(e_{i\pm}^n)|| \) is arbitrarily small. (We find a positive \( g \in K_0(A) \) with \( ||D(g)|| \) sufficiently small and then find projections \( e_{i\pm}^n \) such that \( [e_{i+}^n] = [n, i](g + h_n^0(e_{i}^n)) \) and \( [e_{i-}^n] = [n, i](g) \).) Then by Lemma 3.4 we find a unitary \( w_n \in A \cap B_n^\prime \) such that
\[
[w_n] = 0, \\
B(w_n, z_n) = -[e_{i+}^n] + [e_{i-}^n] = -[n, i]h_n^0(e_{i}^n)
\]
and \( ||[w_n, z_n]|| \) is arbitrarily small for \( i = 1, \ldots, k_n \). Since
\[
B(w_{n+1}^* w_n, z_n) = \sum_j B(w_{n+1}^* p_{n+1, j}, z_n p_{n+1, j})
\]
\[
= \sum_j \chi_{n+1}^1(j, i)[n, i]B([w_{n+1}^* z_n p_{n+1, j}]) / [n + 1, j]
\]
\[
= \sum_j \chi_{n+1}^1(j, i)[n, i]h_n^0(e_{i+1}^n)
\]
\[
= [n, i]h_{n+1}^0 \chi_n^1(e_{i}^n),
\]
we have that
\[
B(w_{n+1}^* w_n, z_n) = 0.
\]
Since \( D(B(w_n, z_n)) = 0 \), we have that \( \tilde{k}_i = 0 \) for \( k_i = 1/2\pi i \log z_n \text{Ad } u_n(z_{ni}^n) \), and hence that \( \tilde{h}_i = 0 \) for \( h_i = 1/2\pi i \log \alpha(z_{ni}) \text{Ad } u_n w_n(z_{ni}^n) \). Thus by replacing \( u_n \) by \( u_n w_n \), we have the conclusion.
Proof. As in the proof of Lemma 3.6 we have to decide

$$K_1(A) \rightarrow K_0(M_{\alpha,n}) \rightarrow K_0(A_n) \rightarrow 0$$

is obtained as the inductive limit of

$$
\begin{array}{cccccccc}
0 & \rightarrow & K_1(A) & \rightarrow & K_0(M_{\alpha,n}) & \rightarrow & K_0(A_n) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & K_1(A) & \rightarrow & K_0(M_{\alpha,n+1}) & \rightarrow & K_0(A_{n+1}) & \rightarrow & 0
\end{array}
$$

By defining a homomorphism $\varphi_n : K_0(A_n) \rightarrow K_0(M_{\alpha,n})$ just as in Lemma 3.3, we identify $K_0(M_{\alpha,n})$ with $K_1(A) \oplus K_0(A_n)$ and find a homomorphism $\psi_n^1 : K_0(A_n) \rightarrow K_1(A)$ as in the following diagram:

$$
\begin{array}{cccccccc}
0 & \rightarrow & K_1(A) & \rightarrow & K_1(A) & \oplus & K_0(A_n) & \rightarrow & K_0(A_n) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \| & & \\
0 & \rightarrow & K_1(A) & \rightarrow & K_1(A) & \oplus & K_0(A_{n+1}) & \rightarrow & K_0(A_{n+1}) & \rightarrow & 0
\end{array}
$$

Lemma 3.8 The homomorphism $\psi_n^1 : K_0(A_n) \rightarrow K_1(A)$ is given by

$$[p_{n,i}] \mapsto [u_{n+1}u_np_{n,i}]$$

where $[p_{n,i}] = [n, i]_e$ with $(e_i)$ the canonical basis for $Z^{n,\pi} = K_0(A_n)$ and $[u_{n+1}u_np_{n,i}]$ is divisible by $[n, i]$. Proof. As in the proof of Lemma 3.6 we have to decide

$$[t \mapsto \text{Ad} v_{n,t}(p_{n,i})] \rightarrow [t \mapsto \text{Ad} v_{n+1,t}(p_{n,i})] \quad (*)$$

in $K_0(M_{\alpha,n+1})$, where $p_{n,i}$ denotes $p_{n,i} \oplus 0$ in $A \otimes M_2$. (Note that $\text{Ad} u_n(p_{n,i}) = \alpha(p_{n,i})$ and $\text{Ad} u_{n+1}(p_{n,i}) = p_{n,i}$.) Note that the identification of $K_1(A)$ with $K_0(SA)$ is done in such a way that $[u_n]$ corresponds to

$$[t \mapsto \text{Ad} v_{n,t}(1 \oplus 0)] - [(1 \oplus 0)]$$

([1] 8.2.2). Since

$$[t \mapsto \text{Ad} v_{n,t}(p_{n,i})] = [t \mapsto \text{Ad} v_{n,t}(1 \oplus 0)] - [t \mapsto \text{Ad} v_{n,t}(1 - p_{n,i})],$$

(*) equals

$$[t \mapsto \text{Ad} v_{n,t}(1 \oplus 0)] - [t \mapsto \text{Ad}(v_{n,t}(1 - p_{n,i}) + v_{n+1,t}p_{n,i})(1 \oplus 0)]$$

$$= [u_n] - [u_n(1 - p_{n,i}) + u_{n+1}p_{n,i}] = [u_{n+1}u_np_{n,i}],$$

where we have used the fact that

$$t \mapsto v_{n,t}((1 - p_{n,i}) \oplus (1 - \alpha(p_{n,i}))) + v_{n+1,t}(p_{n,i} \oplus \alpha(p_{n,i}))$$

is a path of unitaries from $1 \oplus 1$ to

$$(u_n(1 - p_{n,i}) + u_{n+1}p_{n,i}) \oplus (u_n^*(1 - \alpha(p_{n,i})) + u_{n+1}^*\alpha(p_{n,i})).$$

Lemma 3.9 Suppose that $\tilde{\eta}(\alpha) = 0$. Then there is a unitary $u_n \in A$ for each $n$ such that

$$\alpha|B_n = \text{Ad} u_n|B_n,$$

$$\|\alpha(z_m) - \text{Ad} u_n(z_m)\| < 2^{-n}, \quad m \leq n,$$

$$B(u^*_n u_n, z_{ni}) = 0,$$

$$[u_n^* u_n z_{ni}] = 0,$$

$$h_{ni} = 0$$

for $i = 1, \ldots, k_n$, where

$$h_{ni} = \frac{1}{2\pi i} \log \alpha(z_{ni}) \text{Ad} u_n(z_{ni}^*).$$
Proof. Comparing with Lemma 3.7, the newly appeared conditions are only
\[ [u_{n+1}^* u_n p_{ni}] = 0. \]
We will find a unitary \( w_n \in A \cap B'_n \) such that \( [w_n, z_n] = 0 \) and the above conditions are satisfied by replacing all \( u_n \) by \( u_n w_n \). With the condition \( [w_{n+1}, z_{n+1}] = 0 \), it follows that \( [w_{n+1}, z_n] = 0 \) and that the other conditions are preserved.

From the assumption that
\[
0 \rightarrow K_1(A) \rightarrow K_0(M_\alpha) \rightarrow K_0(A) \rightarrow 0
\]
is trivial, we have a homomorphism \( h^1_n : K_0(A_n) \rightarrow K_1(A) \) for each \( n \) such that
\[
\psi_n^1 = h^1_n - h^1_{n+1} \chi_n.
\]
We only have to find a unitary \( w_n \in A \cap B'_n \) such that \( [w_n, z_n] = 0 \) and
\[
[w_n p_{ni}] = -[n, i] h^1_n(e_i), \quad i = 1, \ldots, k_n.
\]
Since \( z_n p_{ni} \) in \( p_{ni} A_m p_{ni} \cap B'_m \) for \( m > n \) is a direct sum of elements of the form
\[
\begin{pmatrix}
0 & z_{n+1}^L p_{ni} \\
1 & \ddots & \ddots \\
& \ddots & \ddots \\
& & 1 & 0
\end{pmatrix}
\]
with \( L = \pm 1 \), this follows immediately.

Proof of (1)\( \Rightarrow \) (2) of Theorem 3.1.

Under the assumption (1) we have found a sequence \( \{u_n\} \) of unitaries as in the previous lemma. Now we apply the homotopy lemma to the pair \( u_{n+1}^* u_n p_{ni}, \ z_n p_{ni} \) of unitaries in \( p_{ni} A_{n+1} p_{ni} \cap B'_ni \) ([4] 8.1): From the conditions
\[
B(u_{n+1}^* u_n, z_n) = 0, \quad [u_{n+1}^* u_n p_{ni}] = 0
\]
calculated in \( p_{ni} A_{n+1} p_{ni} \cap B'_ni \), which follow since \( K_*(p_{ni} A_{n+1} p_{ni} \cap B'_n) \rightarrow K_*(p_{ni} A_{ni} p_{ni}) \rightarrow K_*(A) \) are injective, and the condition \( \|u_{n+1}^* u_n, z_n]\| = 0 \) as \( n \rightarrow \infty \), we obtain a continuous path \( v_{ni,t} \) of unitaries in \( p_{ni} A_{ni} \cap B'_ni \) such that
\[
v_{ni,0} = p_{ni}, \quad v_{ni,1} = u_{n}^* u_{n+1} p_{ni}
\]
and
\[
\max_t \|v_{ni,t}, z_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
Let \( v_{ni} = \sum_t v_{ni,t} \), and define a continuous path \( v_t \) of unitaries for \( t \in [1, \infty) \) by
\[
v_1 = u_1, \quad v_{n+t} = u_n v_{n,t}, \quad 0 \leq t \leq 1
\]
for \( n = 1, 2, \ldots \). Then since
\[
\max_t \|v_{ni,t}, z_m\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
we obtain that for any \( m \),
\[
\lim_{t \rightarrow \infty} \text{Ad} \ v_t(z_m) = \alpha(z_m).
\]
We also have that for \( t \geq m \) and \( a \in B_m \)
\[
\text{Ad} \ v_t(a) = \alpha(a).
\]
Thus it follows that for any \( x \in A \)
\[
\lim_{t \rightarrow \infty} \text{Ad} \ v_t(x) = \alpha(x).
\]
This completes the proof.
4 Main Theorem

Proposition 4.1 If \( \varphi \in \text{Hom}(K_1(A), \text{Aff}(T_A)) \), there exists an automorphism \( \alpha \in \text{Im}(A) \) such that \( \eta(\alpha) \) is trivial and the rotation map \( R_\alpha : K_1(M_n) \rightarrow \text{Aff}(T_A) \) is given by

\[
R_\alpha(a, b) = D(a) + \varphi(b)
\]

for some identification of \( K_1(M_n) \) with \( K_0(A) \oplus K_1(A) \).

To prove this we first prepare:

Lemma 4.2 If \( \varphi \in \text{Hom}(K_1(A), \text{Aff}(T_A)) \), there exists an inductive system

\[
\begin{array}{cccc}
Z^{k_1} & \xrightarrow{\chi_1} & Z^{k_2} & \xrightarrow{\chi_2} & Z^{k_3} & \rightarrow & \ldots
\end{array}
\]

whose limit is isomorphic to \( K_1(A) \) for \( i = 0, 1 \) and homomorphisms \( h_n : Z^{k_n} \rightarrow Z^{k_{n+1}} \) such that

\[
\begin{align*}
|\varphi \circ \chi_{\infty, n-1}^1(e_{i, j}^{n-1}) & - D \circ \chi_{\infty, n}^0 \circ h_{n-1}(e_{i, j}^{n-1})| < 2^{-n+1} \ell_n^{-1} D \circ \chi_{\infty, n-1}^0(e_{i, j}^{n-1}), \\
|h_{n-1} \circ \chi_{\infty, n-2}^1(e_{i, j}^{n-1}) & - \chi_{\infty, n-1}^0 \circ h_{n-2}(e_{i, j}^{n-1})| < 2^{-n+3} \ell_n^{-1} \chi_{\infty, n-2}^0(e_{i, j}^{n-1}),
\end{align*}
\]

where that \( |x| < y \) for \( x, y \in Z^{k_n} \) means that \( |x_i| < y_i \) for all \( i \), \( (e_{i, j}^n)_j \) is the canonical basis for \( Z^{k_n} \),

\[
\ell_n = \max\{|[n, j] | j = 1, \ldots, k_n\},
\]

and \( ([n, j])_j \in Z^{k_n} \) corresponds to \( [1] \in K_0(A) \).

Proof. Suppose that we are given inductive systems

\[
\begin{array}{cccc}
Z^{k_1} & \xrightarrow{\chi_1} & Z^{k_2} & \xrightarrow{\chi_2} & Z^{k_3} & \rightarrow & \ldots
\end{array}
\]

such that the limit is isomorphic to \( K_1(A) \) for \( i = 0, 1 \) and \( \chi_n^0(i, j) \geq 2^{n+1} \max(|\chi_n^1(i, j)|, 1) \). By passing to a subsequence we construct the homomorphisms \( h_n \) with the required properties.

Suppose that we have constructed \( h_1, \ldots, h_{n-1} \) and fixed \( Z^{k_1}, \ldots, Z^{k_n} \). Then we compute \( \ell_n \) and find \( \chi : Z^{k_n} \rightarrow K_0(A) \) such that

\[
|\varphi \chi_{\infty, n}(e_{i, j}^n) - D \chi(e_{i, j}^n)| < 2^{-n+1} \ell_n^{-1} D \chi_{\infty, n}(e_{i, j}^n).
\]

This is obviously possible by the density of Range \( D \) and

\[
\inf_{\tau \in T_A} D \chi_{\infty, n}(e_{i, j}^n)(\tau) > 0.
\]

Then we find an \( m > n \) such that Range \( \chi \subset \text{Range} \chi_{\infty, m} \), and \( \eta : Z^{k_n} \rightarrow Z^m \) such that

\[
\begin{array}{cccc}
Z^{k_n} & \xrightarrow{\chi_{\infty, m}} & \xrightarrow{\eta} & Z^m
\end{array}
\]

is commutative. Note

\[
\begin{align*}
&D \chi_{\infty, m} \eta \chi_{n-1}^1(e_{i, j}^{n-1}) - D \chi_{\infty, n} h_{n-1}(e_{i, j}^{n-1}) \\
&\leq |D \chi_{\infty, m} \eta \chi_{n-1}^1(e_{i, j}^{n-1}) - \varphi \chi_{\infty, n-1}^1(e_{i, j}^{n-1})| + |\varphi \chi_{\infty, n-1}^1(e_{i, j}^{n-1}) - D \chi_{\infty, n} h_{n-1}(e_{i, j}^{n-1})| \\
&< 2^{-n+1} \ell_n^{-1} \sum_{i=1}^{k_n} D \chi_{\infty, m}(e_{i}^{n}) \chi_{n-1}^1(i, j) + 2^{-n+1} \ell_n^{-1} D \chi_{\infty, n-1}(e_{i}^{n-1}) \\
&< 2^{-n+1} \ell_n^{-1} \sum_{i} D \chi_{\infty, m}(e_{i}^{n}) \chi_{n-1}^1(i, j) + 2^{-n+1} \ell_n^{-1} D \chi_{\infty, n-1}(e_{i}^{n-1}) \\
&< (2^{-n+1} + 2^{-n+1} \ell_n^{-1}) D \chi_{\infty, n-1}(e_{i}^{n-1}) \\
&< 2^{-n+2} \ell_n^{-1} D \chi_{\infty, n-1}(e_{i}^{n-1}.
\]

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Thus by choosing a sufficiently large $\ell > m$ it follows that

$$|\chi_{\ell m}^0 \chi_{n-1}^0 (e_j^{n-1}) - \chi_{\ell n}^0 h_{n-1} (e_j^{n-1})| < 2^{-n+2} \ell_{n-1}^0 (e_j^{n-1}).$$

By taking $Z^{k_1}$ for $Z^{k_1+1}$ and $\chi_{\ell m}^0$ for $h_n$, this completes the proof.

**Proof of Proposition 4.1**

By the previous lemma we have the following diagram:

$$\begin{array}{cccc}
\ldots & \longrightarrow & Z^{k_0} & \longrightarrow & Z^{k_{n+1}} & \longrightarrow & \ldots & \longrightarrow & K_1(A) \\
\ldots & \longrightarrow & Z^{k_0} & \longrightarrow & Z^{k_{n+1}} & \longrightarrow & \ldots & \longrightarrow & K_0(A) \\
\end{array}$$

with the specified properties. Accordingly we construct an increasing sequence $\{A_n\}$ of $T$ algebras such that

$$A_n = B_n \otimes C(T),$$

$$B_n = \oplus_{i=1}^n B_{ni},$$

$$B_{ni} \cong M_{[n,i]}$$

and the embeddings of $A_n$ into $A_{n+1}$ are in the standard form. By Elliott’s theory [7], we identify $\bigcup_{n=1}^{\infty} A_n$ with $A$.

Define $\psi_n^0 : K_1(A_n) \to K_0(A_{n+2})$ by

$$\psi_n^0 = h_{n+1} \chi_1^1 - \chi_{n+1}^0 h_n.$$ 

By the properties specified in Lemma 4.2 we have that

$$|\psi_n^0(i,j)| < 2^{-n+1} \ell_n^0 \chi_{n+2,n}^2(i,j).$$

Then by Lemma 3.4 (and its proof) we find a unitary $w_{nj} \in B_{n+2} \cap B_n'$ such that

$$w_{nj} = w_{nj} p_{nj} + 1 - p_{nj},$$

$$\|\text{Ad } w_{nj}(z_{nj}) - z_{nj}\| \leq 3 \pi 2^{-n+1},$$

$$B(w_{nj}, z_{nj}) = -[n, j] \psi_n^0(e_j^0).$$

(Because $z_{nj}$ in $B_{n+2,j} \otimes C(T)$ is a direct sum of elements of the form as in the proof of Lemma 3.4 such that the matrix sizes $M_s$ are at least $2^n$; hence the error introduced by choosing $N_s$ in that proof will be of the order $2^{-2n}$.) If $w_n$ denotes $w_{n1} w_{n2} \cdots w_{nk_n}$, then we have that

$$w_n \in B_{n+2} \cap B_n',$$

$$\|\text{Ad } w_n(z_n) - z_n\| \leq 3 \pi 2^{-n+1},$$

$$B(w_n, z_n) = -[n, j] \psi_n^0(e_j^0),$$

$$[w_n p_{nj}] = 0.$$ 

We define the following two automorphisms $\beta_0, \beta_1$ of $A$ by

$$\beta_0 = \lim_{n \to \infty} \text{Ad}(w_2 w_4 \cdots w_{2n}),$$

$$\beta_1 = \lim_{n \to \infty} \text{Ad}(w_1 w_3 \cdots w_{2n-1}).$$

To show the limits exist, note that $[w_m, w_n] = 0$ if $|m - n| \geq 2$ and the limits obviously exist on $\bigcup_{n=1}^{\infty} B_n$. Since $\text{Ad}(w_n w_{n+2} \cdots w_{n+2k})(z_n)$ in $A_{n+2k+2}$ is a direct sum of elements of the form

$$
\begin{pmatrix}
0 & z_{n+2k+2}^L \\
1 & \\
& \ddots \\
& & 1 & 0
\end{pmatrix}
$$

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with \( L = \pm 1 \), we have that
\[
\| \text{Ad}(w_n \cdots w_{n+2k}w_{n+2k+2})(z_n) - \text{Ad}(w_n \cdots w_{n+2k})(z_n) \| < 3\pi 2^{-(n+2k+1)}.
\]
Then it also follows that the limits exist on \( z_1, z_2, \ldots \). Since the same reasoning applies to the inverses, we have shown that \( \beta_0, \beta_1 \) exist as automorphisms.

Now we shall show that the product \( \beta_0 \beta_1 \) has the required properties.

By [11], 2.4 the extension \( \eta_i(\beta_i) \)
\[
0 \longrightarrow K_1(A) \longrightarrow K_0(M_{\beta_i}) \longrightarrow K_0(A) \longrightarrow 0
\]
is trivial for \( i = 0, 1 \) and the extension \( \eta_0(\beta_i) \)
is given as the inductive limit of
\[
0 \longrightarrow \mathbb{Z}^{k_n} \longrightarrow \mathbb{Z}^{k_n} \oplus \mathbb{Z}^{k_n} \longrightarrow \mathbb{Z}^{k_n} \longrightarrow 0
\]
with \( n \equiv i \pmod{2} \). Hence \( \eta_i(\beta_0 \beta_1) = \eta_i(\beta_0) + \eta_i(\beta_1) = 0 \). We will compute \( \eta_0(\beta_0) + \eta_0(\beta_1) \) below.

Define \( E = \{(x, y) \in K_1(M_{\beta_0}) \oplus K_1(M_{\beta_1}) \mid q(x) = q(y)/(a, -a) \mid a \in K_0(A)\} \).

If \( g \in K_1(A) \) is the image of \( x_{2n+1} \in \mathbb{Z}^{k_{2n+1}} \), define \( \eta_n : \text{Range} \chi_{2n+1} \rightarrow K_1(M_{\beta_0}) \oplus K_1(M_{\beta_1}) \) by
\[
\eta_n(g) = (h_{2n+1}(x_{2n+1}), x_{2n+2}) \oplus (0, x_{2n+1}),
\]
where the right hand side should be regarded as an element of \( K_1(M_{\beta_0}) \oplus K_1(M_{\beta_1}) \). Then
\[
\eta_{n+1}(g) - \eta_n(g)
= (h_{2n+3}(x_{2n+3}) - \psi_{0}^{0}(x_{2n+2}) - \chi_{2n+4,2n+2}^{0}h_{2n+1}(x_{2n+1}), 0)
\oplus (-\psi_{0}^{0}(x_{2n+3}), 0)
= (\chi_{2n+3}^{0}h_{2n+2}(x_{2n+2}) - \chi_{2n+4,2n+2}^{0}h_{2n+1}(x_{2n+1}), 0)
\oplus (-h_{2n+2}(x_{2n+2}) + \chi_{2n+2}^{0}h_{2n+1}(x_{2n+1}), 0).
\]
Thus \( \eta_n(g) \) gives a well-defined homomorphism \( \eta : K_1(A) \rightarrow E \) such that \( q \eta = \text{id} \). This shows that \( \eta_0(\beta_0 \beta_1) = 0 \).

Let \( u_n = w_n w_{n-2} \cdots \). We take a path \( v(t) \) of unitaries in \( A \otimes M_2 \) from \( z_{n_j} \) to \( \beta_0(z_{n_j}) \) by composing the following two paths for even \( m \geq n \):
\[
v_1(t) = R_t(1 \oplus u_m)R_t^{-1}(z_{n_j} \oplus 1)R_t(1 \oplus u_m^*)R_t^{-1}
\]
and a short path \( v_2 \) from \( \text{Ad} u_m(z_{n_j}) \) to \( \beta_0(z_{n_j}) \). For \( \tau \in T_A \) we want to compute
\[
\frac{1}{2\pi i} \int_0^1 \tau(v(t)v(t)^*)dt.
\]
We know the contribution from \( v_1 \) is zero and the contribution from \( v_2 \) is given by
\[
\lim_{k \to \infty} \tau(B(w_{m+2}^*w_{m+4}^* \cdots w_{m+2k}^*, z_{n_j}))
= \lim \tau\left( \sum_{i=1}^{k} \chi_{2m+2i+2}^{0}v_{m+2i}^{0} + \chi_{2m+2i}^{1}(v_{m+2i}^{1}) \right).
\]
Thus we obtain that
\[ R_{\beta_n}([v]) = \sum_{i=1}^{\infty} D\chi_{\infty}^{0} 2m+2i+2\psi_{m+2i}\lambda_{m+2i}^{1} (e_{j}^{n}). \]
A similar computation applies to $\beta_{k}$. For an odd $n$ we let $m = n - 1$ for computing $r_{0} = R_{\beta_{n}}([v])$ and let $m = n$ for computing the corresponding $r_{1}$, and obtain that
\[ r_{0} + r_{1} = \sum_{i=1}^{\infty} D\chi_{\infty}^{0} n+i+2\psi_{n+i}^{0} +\lambda_{n+i}^{1} (e_{j}^{n}) \]
\[ = \sum_{i=1}^{\infty} (D\chi_{\infty}^{0} n+i+2h_{n+i+1}\lambda_{n+i+1}^{1} (e_{j}^{n}) - D\chi_{\infty}^{0} n+i+1h_{n+i+1}\lambda_{n+i+1}^{1} (e_{j}^{n})) \]
\[ = \varphi\chi_{\infty}^{0} (e_{j}^{n}) - D\chi_{\infty}^{0} h_{n+i+1}\lambda_{n+i}^{1} (e_{j}^{n}). \]
Under the identification of $K_{1}(M_{\beta_{k}})$ with $K_{0}(A) \oplus K_{1}(A)$ specified above, the above element corresponds to $(-h_{n+1}\chi_{n}^{0}(e_{j}^{n}), [z_{nj}])$. This implies that $R_{\beta_{0}\beta_{1}}$ satisfies the required properties.

Let $Q$ be the homomorphism of $\text{OrderExt}(K_{1}(A), K_{0}(A))$ into $\text{Ext}(K_{1}(A), K_{0}(A))$ defined by $([E, R]) \mapsto [E]$. Then $\ker Q$ is the subgroup of the isomorphism classes of $(E_{0}, R_{\varphi})$ where $E_{0}$ is the trivial extension $K_{1}(A) \oplus K_{0}(A)$, and $R_{\varphi} : E_{0} \rightarrow \text{Aff}(T_{A})$ is determined by $\varphi \in \text{Hom}(K_{1}(A), \text{Aff}(T_{A}))$ as in the previous proposition:
\[ R_{\varphi} : (a, b) \mapsto D(a) + \varphi(b). \]

**Proposition 4.3** The following sequences of abelian groups are exact:
\[ 0 \rightarrow \ker Q \rightarrow \text{OrderExt}(K_{1}(A), K_{0}(A)) \rightarrow \text{Ext}(K_{1}(A), K_{0}(A)) \rightarrow 0, \]
\[ 0 \rightarrow \text{Hom}(K_{1}(A), \ker D) \rightarrow \text{Hom}(K_{1}(A), K_{0}(A)) \rightarrow \text{Hom}(K_{1}(A), \text{Aff}(T_{A})) \rightarrow \ker Q \rightarrow 0. \]

**Proof.** For the first sequence we only have to show that $Q$ is surjective. Given an extension
\[ 0 \rightarrow K_{0}(A) \rightarrow E \rightarrow K_{1}(A) \rightarrow 0, \]
we regard $K_{0}(A)$ as a subgroup of $E$ and have to extend $D : K_{0}(A) \rightarrow \text{Aff}(T_{A})$ to a homomorphism $R : E \rightarrow \text{Aff}(T_{A})$. This can be done step by step by using the fact that $\text{Aff}(T_{A})$ is divisible.

For the second sequence we only have to show that $(E_{0}, R_{\varphi})$ and $(E_{0}, R_{\psi})$ are isomorphic if and only if $\varphi = \psi + D \circ h$ for some $h \in \text{Hom}(K_{1}(A), K_{0}(A))$. This follows because an isomorphism $\mu : E_{0} \rightarrow E_{0}$ is given by
\[ \mu : (a, b) \mapsto (a + h(b), b) \]
for some $h \in \text{Hom}(K_{1}(A), K_{0}(A))$ with $R_{\psi} \circ \mu = R_{\varphi}$.

**Theorem 4.4** Let $A$ be a simple unital AT algebra of real rank zero, $\text{Imn}(A)$ the group of approximately inner automorphisms of $A$, and $\text{AImn}(A)$ the group of asymptotically inner automorphisms of $A$. Then $\text{AImn}(A)$ is a normal subgroup of $\text{Imn}(A)$ and the quotient $\text{Imn}(A)/\text{AImn}(A)$ is isomorphic to $\text{OrderExt}(K_{1}(A), K_{0}(A)) \oplus \text{Ext}(K_{0}(A), K_{1}(A))$ with isomorphism induced by $\tilde{\eta}$.

**Proof.** Before Theorem 3.1 we have described the homomorphism
\[ \tilde{\eta} : \text{Imn}(A) \rightarrow \text{OrderExt}(K_{1}(A), K_{0}(A)) \oplus \text{Ext}(K_{0}(A), K_{1}(A)), \]
and showed in 3.1 that $\ker \tilde{\eta} = \text{AImn}(A)$. By 3.1 of [11] we have shown that $\eta = (\tilde{\eta}_{0}, \eta_{1}) = (Q\tilde{\eta}_{0}, \eta_{1})$ is surjective onto $\text{Ext}(K_{1}(A), K_{0}(A)) \oplus \text{Ext}(K_{0}(A), K_{1}(A))$. By Proposition 4.1 we know that $\text{Range} \tilde{\eta}$ contains $\ker Q$, which shows that $\tilde{\eta}$ is surjective. This completes the proof.
Example 4.5 If $A$ is the irrational rotation $C^*$-algebra generated by unitaries $u, v$ with $uv^*v^* = e^{2\pi i \theta} 1$ for some irrational number $\theta \in (0, 1)$, then $A$ is a simple unital AT algebra of real rank zero by [9], and $K_i(A) \cong \mathbb{Z}^2$ and hence $\text{Ext}(K_1(A), K_{i+1}(A)) = 0$. But since $A$ has only one tracial state and $\text{Range } D = \mathbb{Z} + \theta \mathbb{Z}$, it follows that $\text{Hom}(K_1(A), \text{Aff}(T_A)) \cong \mathbb{R}^2$ and $\text{OrderExt}(K_1(A), K_0(A)) \cong \mathbb{R}^2/(\mathbb{Z} + \theta \mathbb{Z})^2$ which is isomorphic to $\text{Inn}(A)/\text{Aln}(A)$. Note also that $H\text{Inn}(A) = \text{Inn}(A)$ in this case since the natural $\mathbb{T}^2$ action on $A$ exhausts all $\text{OrderExt}$.

References


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