Crossed products of Cuntz algebras by quasi-free automorphisms

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1 Introduction

By the recent classification theorems of Kirchberg and Phillips [15, 22], a certain class of purely infinite simple C*-algebras is now classified by K-theoretic data. This class includes inductive limits of direct sums of matrix algebras over Cuntz algebras as well as crossed products by certain trace scaling automorphisms (see [24, 25, 20, 22]). We study certain automorphism groups of the Cuntz algebra and show that the resulting crossed products fall in this class.

In Section 2 we study quasi-free actions of countable torsion-free abelian groups on $O_2$: the main result here is that such an action satisfies a certain Rohlin type condition. To prove the first result we study Rohlin properties for several commuting automorphisms by using the CAR algebra formulation [6]. It follows that the crossed product of $O_2$ by a countable torsion-free abelian group with quasi-free action is isomorphic to $O_2$ (this also follows from Lemma 10 and the classification results of [15, 22]).

In Section 3 we consider certain simple crossed products of $O_n$ by quasi-free actions of $\mathbb{R}$. A natural dichotomy results from the condition on the parameters which guarantees simplicity of the crossed product (see [17]). In the case that parameters have the same sign, it was shown in [19] that the crossed product has a unique trace (up to scalar multiple) and is stably projectionless. Otherwise the parameters generate $\mathbb{R}$ as a closed subsemigroup; in this case it is shown in Theorem 2 that the crossed product is stable and purely infinite.

In Proposition 3 we note that simple crossed products of unital purely infinite C*-algebras by $\mathbb{R}$ are stable whether they are infinite or not and in proposition 4 we show that they are either projectionless or traceless. In the course of the proof of the result described in the preceding paragraph we show in Lemma 10 that the reduced crossed product of a purely infinite simple C*-algebra by a discrete group acting by outer automorphisms (for nonidentity elements) is purely infinite (cf. [13]).

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2 Crossed products of $O_2$

The Cuntz algebra $O_n$ with $n$ integer $\geq 2$ is the universal C*-algebra generated by $n$ isometries $S_j$, $j = 1, ..., n$ satisfying

$$\sum_{j=1}^{n} S_j S_j^* = 1.$$ 

It is known [7] that $O_n$ is simple, and hence that it has an action $\alpha$ of $U(n)$ on $O_n$ such that

$$\alpha_g(S_i) = \sum_{j=1}^{n} g_{ji} S_j.$$ 

We call this action or its restriction to a subgroup quasi-free [9].
**Theorem 1** Let $\alpha$ be the action of $T^2$ on the Cuntz algebra $O_2$ defined by
\[
\alpha_z(S_i) = z_i S_i \quad i = 1, 2
\]
for $z = (z_1, z_2) \in T^2$. Let $G$ be a torsion-free subgroup of $T^2$. Then $O_2$ has a central net $\{(S_{\nu 1}, S_{\nu 2})\}$ of pairs of isometries such that
\[
S_{\nu 1} S_{\nu 1}^* + S_{\nu 2} S_{\nu 2}^* = 1,
\]
\[
\lim_\nu \|\alpha_g(S_{\nu i}) - S_{\nu i}\| = 0
\]
for any $g \in G$ and $i = 1, 2$.

**Remark 1** It follows from the above theorem that if $G$ is countable then $A = O_2 \times_{\alpha|G} G$ is isomorphic to $A \otimes O_2$ by Elliott’s result [26] and that $A$ is isomorphic to $O_2$ by Kirchberg [15]. This result was also shown by Rørdam [27] recently, who actually proved it for any outer action.

**Remark 2** The conclusion of the above remark can be proved directly (without appealing to Kirchberg’s result etc.) if $G = \mathbb{Z}$, by proving the uniqueness and existence theorems for unital homomorphisms of $O_2 \times \mathbb{Z}$ into itself or $O_2$ following the methods developed in [4] with Proposition 1 below, where such theorems are proved when the action is trivial.

**Remark 3** If $G$ is a non-trivial torsion subgroup of $\{(z, z); z \in T\} \subset T^2$, then $O_2 \times G$ is not isomorphic to $O_2$ because their $K$-groups are different [10].

For the proof of the above theorem we will need the Rohlin towers with additional properties for the shift of the UHF algebra $M_{2^\infty}$ (cf. [6, 25]).

**Proposition 1** Let $\mu_i \in T$ for $i = 0, \ldots, n$ and let $\gamma_i$ be the automorphism of the UHF algebra $M_{2^\infty}$ defined by
\[
\bigotimes_1^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \mu_i \end{pmatrix}.
\]
Let $\sigma$ be the one-sided shift of $M_{2^\infty}$. Then for any $\epsilon > 0$ and $N \in \mathbb{N}$, there is an orthogonal family
\[
\{e_{j,k}; k = 0, \ldots, N - 1 \text{ for } j = 1, k = 0, \ldots, N \text{ for } j = 2\}
\]
of projections of $M_{2^\infty}$ such that
\[
\sum_j \sum_k e_{j,k} = 1,
\]
\[
\|\sigma(e_{j,k}) - e_{j,k+1}\| < \epsilon,
\]
\[
\|\gamma_i(e_{j,k}) - e_{j,k}\| < \epsilon
\]
for $j = 1, 2$ with $e_{1,N} = e_{1,0}$, $e_{2,N+1} = e_{2,0}$.

**Remark 4** In the above proposition if $\gamma_0^{m_0} \circ \cdots \circ \gamma_n^{m_n} \neq id$ for any $m \in \mathbb{Z}^{n+1} \setminus \{0\}$, one can obtain, for any $N \in \mathbb{N}$, one tower of projections of length $2^N$ with obvious properties as above (instead of two for $j = 1, 2$). The following is also true as seen from the proof of Proposition 1: For any $i = 0, \ldots, n$ there exists an orthogonal family $\{e_k; k = 0, \ldots, 2^N - 1\}$ of projections such that
\[
\sum e_k = 1,
\]
\[
\|\sigma(e_k) - e_k\| < \epsilon, \quad \|\gamma_j(e_k) - e_k\| < \epsilon \text{ for } j \neq i,
\]
\[
\|\gamma_i(e_k) - e_{k+1}\| < \epsilon
\]
with $e_{2^N} = e_0$. 
Let $G$ be the subgroup of the automorphism group of $M_{2^n}$ generated by $\gamma_0, \ldots, \gamma_n$. Then $G$, as a finitely generated abelian group, is isomorphic to $\mathbb{Z}^k \times F$ where $k$ is a non-negative integer and $F$ is a finite group. Since $G \subseteq T$, $F$ must be a cyclic group. Thus in the following we may assume that $\mu_0$ has finite order, i.e., $\mu_0^p = 1$ for some $p > 0$ and $\mu_m = \mu_1^{m_1} \cdots \mu_n^{m_n} \neq 1$ for any $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \setminus \{0\}$.

We use the CAR algebra formulation [6, 5]. Let $A$ be the CAR algebra over the Hilbert space $l^2(\mathbb{N})$, i.e., the $\mathcal{C}^*$-algebra generated by $a(f)$, $f \in l^2(\mathbb{N})$ satisfying the anti-commutation relations. For each isometry $U$ one defines an endomorphism of $A$ by

$$\alpha_U(a(f)) = a(Uf), \ f \in l^2(\mathbb{N}).$$

We denote by $A^e$ the even part of $A$:

$$A^e = \{x \in A; \alpha_{-1}(x) = x\}.$$

Let $\gamma$ be the action of $T$ on $M_{2^n}$ defined by

$$\gamma_{\mu} = \bigotimes_1^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}.$$

Then one has an isomorphism $\Psi$ of $A$ onto $M_{2^n}$ such that

$$\Psi \circ \alpha_V|A^e = \sigma \circ \Psi|A^e,$$

$$\Psi \circ \alpha_{\mu_1} = \gamma_{\mu} \circ \Psi.$$

**Lemma 1** Let $\delta \in (0, 1/2)$ and let $\{\xi_k\}$ and $\{\eta_k\}$ be orthonormal families of vectors in a Hilbert space such that

$$\|\xi_k - \eta_k\| < \delta 2^{-k}, \ k = 1, 2, \ldots.$$

Then there is a unitary operator $W$ such that

$$W^\dagger \eta_k = \xi_k, \ k = 1, 2, \ldots,$$

$$\|W - 1\|_1 < 3\delta,$$

where $\|\cdot\|_1$ is trace norm.

**Proof.** Let $H_{\xi}$ (resp. $H_{\eta}$) be the closed subspace spanned by $\xi_k$ (resp. $\eta_k$), and let $P_{\xi}$ (resp. $P_{\eta}$) be the projection onto $H_{\xi}$ (resp. $H_{\eta}$). Then

$$\|(P_{\xi} - P_{\eta}P_{\xi}P_{\xi})^{1/2}\|_1 \leq \sum_k \|(1 - P_{\eta})\xi_k, \xi_k\|^{1/2} \leq \sum_k (1 - |\xi_k, \eta_k|^2)^{1/2} \leq \sum_k \delta 2^{-k} < \delta,$$

and by symmetry,

$$\|(P_{\eta} - P_{\eta}P_{\xi}P_{\eta})^{1/2}\|_1 < \delta.$$

In particular $\|P_{\eta}(1 - P_{\xi})\| \leq \|P_{\eta}(1 - P_{\xi})\|_1 < \delta$. Let $U_1$ be the partial isometry part of the polar decomposition of $(1 - P_{\xi})(1 - P_{\eta})$:

$$U_1 = a^{-1}(1 - P_{\xi})(1 - P_{\eta})$$
where \( a^{-1} \) is the inverse of \( a = ((1 - P_\xi)(1 - P_\eta)(1 - P_\xi))^{1/2} \) on the range of \((1 - P_\xi)\). Then
\[
\|U_1 - (1 - P_\eta)\|_1 \leq \|a^{-1} - (1 - P_\xi)\|_1 + \|(1 - P_\xi)(1 - P_\eta) - (1 - P_\eta)\|_1 \\
\leq \|a^{-1} - (1 - P_\xi)\|_1 + \delta.
\]
Since \( a \leq 1 - P_\xi \) and
\[
a = (1 - P_\xi - (1 - P_\xi)P_\eta(1 - P_\xi))^{1/2} \geq (1 - \|P_\eta(1 - P_\xi)\|^2)^{1/2}(1 - P_\xi),
\]
one has
\[
\|a^{-1}\| \leq (1 - \delta^2)^{-1/2} < 2,
\]
\[
\|a^{-1} - (1 - P_\xi)\|_1 \leq \|a^{-1}\| \|1 - P_\xi - a\|_1 \\
< 2\|1 - P_\xi - a^2\| < 2\|(1 - P_\xi)P_\eta\|_1 < 2\delta.
\]
Hence \( \|U_1 - (1 - P_\eta)\|_1 < 2\delta \).

Next define a partial isometry \( U_2 \) of \( H_\eta \) onto \( H_\xi \) by
\[
U_2\eta_k = \xi_k.
\]
Then it follows that
\[
\|U_2 - P_\eta\|_1 = \|[(U_2 - P_\eta)^*(U_2 - P_\eta)]^{1/2}\|_1 \\
\leq \sum \|a^{-1} - (1 - P_\xi)\|_1 \|\xi_k\| < \delta.
\]
Thus \( W = U_1 + U_2 \) satisfies the desired properties.

**Lemma 2** Let \( \{\xi_i\} \) be a sequence in \( T^n \) and \( \{k_m\} \) a sequence of positive integers such that for any continuous function \( f \) on \( T^n \),
\[
\lim_{m \to \infty} \frac{1}{k_m} \sum_{i=1}^{k_m} f(\xi_i) = \int_{T^n} f(\xi) \, d^\mu \xi
\]
where \( d^n \xi \) is normalized Haar measure on \( T^n \). Then for any \( \epsilon > 0 \) and \( \lambda \in T^n \) there exists an \( m_0 \) such that for any \( m \geq m_0 \) there is a bijection \( \phi_m \) on \( \{1, 2, \ldots, k_m\} \) with
\[
d(\xi_{\phi_m(i)}, \lambda \xi_i) < \epsilon
\]
where for \( \mu, \nu \in T^n \),
\[
d(\mu, \nu) = \max_i |\mu_i - \nu_i|.
\]

**Proof.** The proof is about the same as in [5].

For a sufficiently large integer \( N \) we partition \( T^n \) into \( N^n \) subsets: For \( k \in \{0, \ldots, N - 1\} \), let
\[
R_k = \{(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \mid \theta_j \in \left[\frac{k_N}, \frac{k_N + 1}{N}\right), j = 1, \ldots, n\}.
\]
For any union \( V \) of subsets \( R_k \) one defines
\[
s(V) = \bigcup \{R_k \mid R_k \bigcap V^c = \emptyset\}
\]
\[
l(V) = \bigcup \{R_k \mid R_k \bigcap V \neq \emptyset\}.
\]
Then \( s(V) \subset s(V) \subset V^c \subset V \subset \bigcap V^c \subset l(V) \subset l(V) \) where \( V^c \) denotes the interior of \( V \) etc. Hence there are continuous functions \( f_V, g_V \) on \( T^n \) such that \( 0 \leq f_V \leq 1, 0 \leq g_V \leq 1, \supp f_V \subset V, \supp g_V \subset l(V) \), and
\[
f_V = 1 \text{ on } s(V), \ g_V = 1 \text{ on } V.
\]
Then it follows that
\[
\sum_{1}^{k_m} f_V(\xi_i) \leq \sharp\{i; 1 \leq i \leq k_m, \xi_i \in V\} \\
\leq \sum_{1}^{k_m} g_V(\xi_i)
\]
and there is an \(m_V\) such that for any \(m \geq m_V\)
\[
|s(V)| \leq \frac{1}{k_m} \sum_{1}^{k_m} f_V(\xi_i) \leq \frac{1}{k_m} \sum_{1}^{k_m} g_V(\xi_i) \leq |l(V)|
\]
where \(\sharp\) means the number of elements of and \(|\cdot|\) means the volume of. (Note that except for the case \(V = T^2\) one can assume the strict inequalities.) Let \(m_0 = \max_V m_V\). Then it follows that for any \(m \geq m_0\) and any \(V\),
\[
|s(V)| \leq \frac{1}{k_m} \sharp\{i; 1 \leq i \leq k_m, \xi_i \in V\} \leq |l(V)|.
\]
Let \(\lambda \in T^n\). For each finite subset \(F\) of \(\{1, \ldots, k_m\}\) define
\[
\phi(F) = \{i; 1 \leq i \leq k_m, d(\xi_i, \lambda \eta_j) \leq 5/N\text{ for some } j \in F\}.
\]
Then we assert that
\[
\sharp\phi(F) \geq \sharp F.
\]
To prove this let \(V\) (resp. \(W\)) be the union of \(R_k\) such that \(R_k \ni \xi_j\) for some \(j \in F\) (resp. \(R_k \ni \xi, d(\xi, \lambda \eta_j) \leq 4/N\) for some \(\xi \in T^n, l \in F\). Then
\[
k_m^{-1}\sharp F \leq k_m^{-1}\sharp\{i; 1 \leq i \leq k_m, \xi_i \in V\} \leq |l(V)|
\]
and
\[
k_m^{-1}\sharp\phi(F) \geq k_m^{-1}\sharp\{i; 1 \leq i \leq k_m, \xi_i \in W\} \geq |s(W)|.
\]
If \(\xi \in l(V)\) then \(d(\xi, \xi_j) \leq 2/N\) for some \(j \in F\). Then \(d(\lambda \xi, \lambda \xi_j) \leq 2/N\) and so \(\lambda \xi \in s(W)\). Hence \(\lambda \cdot l(V) \subset s(W)\), which implies that \(|l(V)| \leq |s(W)|\). Hence one has shown that \(\sharp\phi(F) \geq \sharp F\). Then by the matching theorem as in [5] it follows that there is a bijection \(\phi_m\) of \(\{1, \ldots, k_m\}\) onto itself such that \(\phi(j) \in \phi(\{j\})\), i.e.,
\[
d(\xi_{\phi_m(j)}, \lambda \xi_j) \leq 5/N.
\]
This concludes the proof.

**Lemma 3** Suppose that \(\mu_0\) has finite order and that \(\mu^m \neq 1\) for any \(m \in \mathbb{Z}^n \setminus \{0\}\). Then for any \(\epsilon > 0\) and \(\lambda \in T^n\) there is a unitary \(u \in A^\circ\), the even part of the CAR algebra, such that
\[
||\alpha_V(u) - u|| < \epsilon,  \\
\alpha_{\mu_0}(u) = u,  \\
||\alpha_{\mu_i}(u) - \lambda_i u|| < \epsilon, \ i = 1, \ldots, n.
\]

**Proof.** The proof is about the same as in [5].

For any sufficiently small \(\delta > 0\) there is an orthonormal family \(\{\xi_k\}\) of vectors of \(l^2(\mathbb{N})\) such that
\[
||V\xi_k - \xi_k|| < 2\delta^{-k},
\]
where \(V\) is the shift operator. Then by Lemma 1 there is a unitary operator \(W\) on \(l^2(\mathbb{N})\) such that \(W - 1\) is of trace class with \(\|W - 1\|_1 \leq 3\delta\), and \(WV\xi_k = \xi_k\), \(k = 1, 2, \ldots\). Let \(K_m\) be the linear subspace spanned by \(\xi_1, \ldots, \xi_m\) and let \(A(K_m)\) be the \(C^*\)-algebra generated by \(a(\xi), \ \xi \in K_m\). Then \(\alpha_V A(K_m) = \alpha_W A(K_m)\)
and $\alpha_{W^*}$ is close to the identity within $||W - 1||_1$. We may suppose that the order $p$ of $\alpha_{\nu_0}$ is even. Thus it suffices to find a unitary $u$ in $A(K_m)^{\alpha_{\nu_0}}$ for some $m$ such that

$$||\alpha_{\mu_i}(u) - \lambda_i u|| < \epsilon, \quad i = 1, \ldots, n.$$ 

More concretely let $\{\xi_k\}$ be the sequence in $\mathbf{T}^n$ defined as follows: If $k = \sum_{j=0}^{m-1} a_j 2^j$, let $d(k) = \sum_{j=0}^{m-1} a_j$ and let

$$\xi_k = (\mu_1^{d(k)}, \ldots, \mu_n^{d(k)}).$$

Then $\{\xi_k\}_{0 \leq k < 2^m}$ is the sequence of joint eigenvalues of $(U_1(m), \ldots, U_n(m))$ where

$$U_j(m) = \bigotimes_1^m \begin{pmatrix} 1 & 0 & \mu_j \\ 0 & 1 & 0 \end{pmatrix}.$$

Let for $j = 0, \ldots, p - 1$,

$$X_j(m) = \{k; \ 0 \leq k < 2^m, \ d(k) = j \bmod p\}$$

Then $\{\xi_k; \ k \in X_j(m)\}$ is the sequence of the eigenvalues of

$$(U_1(m), \ldots, U_n(m))$$

on the eigenspace of $U_0(m)$ corresponding to the eigenvalue $\mu_0^j$. We have to show that

$$\lim_{m \to \infty} \frac{1}{\sharp X_j(m)} \sum_{k \in X_j(m)} f(\xi_k) = \int_{\mathbf{T}^n} f(\xi) d^n\xi.$$

It suffices to show that for $f(z) = z_1^{l_1} \cdots z_n^{l_n}$ with $l \in \mathbb{Z}^n \setminus \{0\}$,

$$Y_j(m) = 2^{-m} \sum_{k \in X_j(m)} f(\xi_k)$$

goest to zero as $m \to \infty$. Here we have used that $2^{-m} \sharp X_j(m) \to 1/p$ as $m \to \infty$. Since

$$X_j(m + 1) = X_j(m) \bigcup \{k + 2^m; \ k \in X_{j-1}(m)\}$$

it follows that

$$\begin{pmatrix} Y_0(m+1) \\ \vdots \\ Y_{p-1}(m+1) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & \cdots & 0 & \rho \\ \rho & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \rho \\ Y_{p-1}(m) \end{pmatrix} \begin{pmatrix} Y_0(m) \\ \vdots \\ \vdots \\ Y_{p-1}(m) \end{pmatrix}$$

where $\rho = \mu_1^{l_1} \cdots \mu_n^{l_n} \in \mathbf{T} \setminus \{1\}$. The eigenvalues of the above matrix are

$$1 + \rho e^{-2\pi i j/p}; \ j = 0, \ldots, p - 1$$

with modulus less than 2, which implies that $Y_j(m) \to 0$. Then by using Lemma 2 one can conclude the proof. For example, to a bijection $\phi$ of $X_j(m)$ one associates a unitary $u$ of the direct summand of $A(K_m)^{\alpha_{\nu_0}}$ corresponding to the eigenvalue $\mu_0^j$ of $U_0(m)$ such that $u$ sends the eigenspace of $(U_1(m), \ldots, U_n(m))$ corresponding to $k$ to the one corresponding to $\phi(k)$ for each $k \in X_j(m)$. If $d(\phi(k)), \lambda_k) < \epsilon$ for any $k \in X_j(m)$, then it follows that $||\alpha_{\mu_i}(u) - \lambda_i u|| < 2\pi \epsilon$.

**Lemma 4** Let $\{k_m\}$ be an increasing sequence of positive integers and $\{\xi_m\}, \{\eta_m\}$ sequences of maps of $\{1, \ldots, k_m\}$ into $\mathbf{T}$ such that for any continuous function $f$ on $\mathbf{T}$,

$$\lim_{k_m \to \infty} \frac{1}{k_m} \sum_{i=1}^{k_m} f(\xi_m(i)) = \int_{\mathbf{T}} f(\xi) d\xi$$
and the same equality holds with \( \eta_m \) in place of \( \xi_m \). Then for every \( \epsilon > 0 \) there is an \( m_0 \) such that for any \( m \geq m_0 \) there is a bijection \( \phi \) of \( \{1, \ldots, k_m\} \) onto itself such that
\[
d(\xi_m(i), \eta_m(\phi(i))) < \epsilon
\]
for \( i = 1, \ldots, k_m \).

**Proof.** The proof is about the same as the proof of Lemma 2. See also the proof of Lemma 4.2 in [5].

**Lemma 5** For \( \mu \in \mathbf{T} \) let \( \gamma_\mu \) be the automorphism of \( M_{2\infty} \) defined by
\[
\bigotimes_l \text{Ad} \left( \begin{array}{cc} 1 & 0 \\ 0 & \mu \end{array} \right).
\]
Let \( \rho \in \mathbf{T} \) be such that \( \rho^k \neq 1 \) for any \( k \neq 0 \) and let \( \rho_0 \in \mathbf{T} \) be such that \( \rho_0^p = 1 \) for some \( p > 0 \). Then \( \gamma_\mu(M_{2\infty})^\rho_0 \) satisfies the Rohlin property, i.e., for any \( N \in \mathbf{N} \) and \( \epsilon > 0 \) there is an orthogonal family
\[
\{e_{j,k}; k = 0, \ldots, N - 1 \text{ for } j = 1, k = 0, \ldots, N \text{ for } j = 2\}
\]
of projections of the fixed point algebra of \( M_{2\infty} \) under \( \gamma_\rho_0 \) such that
\[
\sum_j \sum_k e_{j,k} = 1,
\]
\[
\|\gamma_\rho(e_{j,k}) - e_{j,k+1}\| < \epsilon
\]
for \( j = 1, 2 \) with \( e_{1,N} = e_{1,0} \), \( e_{2,N+1} = e_{2,0} \).

**Proof.** The proof of Lemma 3 shows that the eigenvalues of
\[
U_\rho(m) = \bigotimes_l \left( \begin{array}{cc} 1 & 0 \\ 0 & \rho \end{array} \right)
\]
on each eigenspace of \( U_\rho_0(m) \) satisfy the assumption of Lemma 4 as \( m \to \infty \). Also when \( k_m \) is the dimension of such an eigenspace, define non-negative integers \( q, r \) by \( k_m = qN + r, \ 0 < r \leq N \), and then the sequence of length \( k_m \) in \( \mathbf{T} \) consisting of:
\[
\exp(2\pi i k \frac{N(q-r)}{N(q-r)}), \ 0 \leq k < N(q-r); \ \exp(2\pi i l \frac{1}{(N+1)r}), \ 0 \leq l < (N+1)r
\]
satisfies the assumption as \( m \to \infty \). For the adjoint action of a unitary with the latter sequence as eigenvalues, it is easy to construct a family of projections satisfying the condition of the lemma known as the Rohlin property [6, 5]. Thus by applying Lemma 4 one obtains the desired projections.

**Proof of Proposition 1.** Let \( V \) be the shift on \( l^2(\mathbf{N}) \) as before and we are assuming that \( \mu_0 \) has finite even order \( p > 0 \) and that \( \mu_0^m \neq 1 \) for any \( m \in \mathbf{Z}^n \ \backslash \{0\} \). Let \( \rho \in \mathbf{T} \) be such that \( \rho_k \neq 1 \) for any \( k \neq 0 \). Then by Lemma 5 with \( \rho_0 = \mu_0 \) one obtains a family \( \{e_{j,k}\} \) of projections in \( (M_{2\infty})^\rho_0 \) satisfying the Rohlin property. We may assume that all these projections lie in the fixed point algebra of
\[
M_{2M} \otimes 1 \ (\subset M_{2\infty})
\]
for some \( M \in \mathbf{N} \). We choose \( \delta > 0 \) sufficiently small for this \( M \) (specified below). Let \( \{\xi_k; k = 1, \ldots, M\} \) be an orthonormal set of vectors in \( l^2(\mathbf{N}) \) such that
\[
\|V\xi_k - \rho\xi_k\| < \delta/2.
\]
Then by Lemma 3 there is a commuting family \( \{ u_k; \ k = 1, \ldots, M \} \) of unitaries of \( A^\alpha_{\mu_0} \) in the commutant of the finite dimensional C\(^*\)-subalgebra generated by \( a(\xi_k), \ k = 1, \ldots, M \) such that

\[
\| \alpha_V(u_k) - u_k \| < \delta/2, \\
\| \alpha_{\mu_i}(u_k) - \mu_i u_k \| < \delta.
\]

(Since \( A \) is a UHF-algebra we can easily put the additional property of commutativity on \( \{ u_k \} \).) Let

\[
a_k = a(\xi_k)u_k, \ k = 1, \ldots, M.
\]

Then they satisfy the anti-commutation relations and that

\[
\| \alpha_{\mu_i}(a_k) - a_k \| < \delta, \| \alpha_V(a_k) - \rho a_k \| < \delta.
\]

The C\(^*\)-subalgebra \( B \) generated by these \( a_k \) is isomorphic to \( M_{2^M} \), and for a sufficiently small \( \delta > 0 \), \( \alpha_{\mu_i}, \ 1 \leq i \leq n \), are close to the identity on \( B \), \( \alpha_V \) is close to \( \gamma_\rho \) on \( B \), and \( \alpha_{\mu_0} \) is exactly \( \gamma_{\mu_0} \) on \( B \) (by using the identification with \( M_{2^M} \)). Then taking \( e_{i,j} \) of \( B \) using \( B \cong M_{2^M} \), one concludes the proof.

**Proposition 2** For \( i = 1, \ldots, n \) let \( \gamma_i \) be the automorphism of the UHF algebra \( M_{3^n} \) defined by\( \bigotimes_i \ Ad \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \mu_{i1} & 0 \\ 0 & 0 & \mu_{i2} \end{array} \right) \). Suppose that \( \gamma^m = \gamma_i^{m_1} \cdots \gamma_j^{m_n} \neq \text{id} \), i.e., \( (\prod_i^{m_i} \mu_{i1}^{-1} \cdot \prod_j^{m_j} \mu_{j2}^{-1}) \neq (1,1) \) for any \( m \in \mathbb{Z}^n \setminus \{0\} \). Then for any \( \epsilon > 0 \) and \( \lambda \in \mathbb{T}^n \) there is a unitary \( u \in M_{3^n} \) such that

\[
\| \gamma_i(u) - \lambda_i u \| < \epsilon, \ i = 1, \ldots, n.
\]

**Proof.** The proof is about the same as the proof of Lemma 3. We only have to show that the joint eigenvalues of commuting unitaries \( (U_1(m), \ldots, U_n(m)) \) are uniformly distributed as \( m \to \infty \) where

\[
U_j(m) = \bigotimes_i^{m} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \mu_{i1} & 0 \\ 0 & 0 & \mu_{i2} \end{array} \right).
\]

Define a sequence \( \{ \xi_k \} \) in \( \mathbb{T}^n \) as follows: If \( k = \sum_{j=0}^{m-1} a_j 3^j \), let \( d_i(k) = \# \{ j; \ a_j = i \} \) for \( i = 1,2 \) and

\[
\xi_k = (\mu_{i1}^{d_i(k)} \mu_{i2}^{d_i(k)}, \ldots, \mu_{n1}^{d_i(k)} \mu_{n2}^{d_i(k)}).
\]

Then we see that \( \{ \xi_k \}_{0 \leq k < 3^m} \) is the sequence of joint eigenvalues of

\[
(U_1(m), \ldots, U_n(m))
\]

and prove that for \( f(z) = z_1^{l_1} \cdots z_n^{l_n} \) with \( l \in \mathbb{Z}^n \setminus \{0\} \)

\[
3^{-m} \sum_{k=0}^{3^m-1} f(\xi_k) = (\frac{1}{3} (1 + \mu_{i1}^{l_1} \cdots \mu_{n1}^{l_n} + \mu_{i2}^{l_1} \cdots \mu_{n2}^{l_n}))^m
\]

which goes to zero as \( m \to \infty \).

**Proof of Theorem 1.** Let \( g_i \in G, i = 1, \ldots, n \) such that \( g_1, \ldots, g_n \) generate a subgroup isomorphic to \( \mathbb{Z}^n \) and set \( \alpha_i = \alpha_{g_i} \). We construct a pair \( (S'_1, S'_2) \) of isometries of \( O_2 \) such that

\[
\alpha_i(S'_j) \approx S'_j, \quad S'_j S_k \approx S_k S'_j
\]

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for $i = 1, \ldots, n$ and $j, k = 1, 2$. Let $\mu_{i1} = g_{i1}, \mu_{i2} = g_{i1}g_{i2}$, and $\mu_{i3} = g_{i2}^2$. Let $\gamma_i$ be the automorphism of $M_{3^\infty}$ defined by

$$\bigotimes_i \text{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{i2}^{-1} & 0 \\ 0 & 0 & \mu_{i3}^{-1} \end{pmatrix}$$

for $i = 1, \ldots, n$. Note that for any $m \in \mathbb{Z}^n \setminus \{0\}$,

$$\prod_i g_{i2}^{m_i} \cdot \prod_i g_{i2}^{2m_i} \cdot g_{i1}^{-m_i} \neq (1, 1).$$

Thus by Proposition 2 for any $\epsilon > 0$ one finds unitaries $u_1, u_2 \in M_{3^\infty}$ such that

$$\|\gamma_k(u_i) - g_{k_i}u_i\| < \epsilon/2$$

for $k = 1, \ldots, n$, $i = 1, 2$. Let $\Phi$ be the endomorphism of $O_2$ defined by

$$\Phi(x) = \sum_{i=1}^2 S_i x S_i^*, \ x \in O_2.$$

There is a $\delta > 0$ smaller than $\epsilon$ such that for any unital homomorphism $\psi$ of $O_3$ into $O_2$ such that for $i = 1, 2, 3$,

$$\|\alpha_k(\psi(S_i)) - \mu_{ki}\psi(S_i)\| < \delta,
\|\Phi(\psi(S_i)) - \psi(S_i)\| < \delta,$$

it follows that

$$\|\alpha_k(\psi(u_j)) - \gamma_{k_j}\psi(u_j)\| < \epsilon/2,
\|\Phi(\psi(u_j)) - \psi(u_j)\| < \epsilon/2.$$

Let

$$V = \sum_{i,j=1}^2 S_i S_j S_i^* S_j^* \in M_{2^\infty} \subset O_2.$$

Note that $\alpha_i(V) = V$ for any $i$ and that $\alpha_i \circ \Phi = \Phi \circ \alpha_i$. By inspecting the proof of stability for $\Phi$ in [24], one finds a unitary $U \in M_{2^\infty} \subset O_2$ such that

$$\|\alpha_i(U) - U\| < \delta/2, \ i = 1, \ldots, n,
\|V - U^* \Phi(U)\| < \delta/2.$$

Because, first for a sufficiently large $m$ one expresses $v = V \Phi(V) \cdots \Phi^{m-1}(V)$ as a product of $m$ unitaries within $2\pi/m$ of 1 which are $\alpha_i$-invariant. Since $v$ belongs to the finite-dimensional subalgebra $M_{2^{m+1}} \otimes 1$ of $M_{2^\infty}$, this can be done. Second we use Rohlin towers (in $M_{2^\infty}$) which are almost $\alpha_i$-invariant, given by Proposition 1. It then follows that

$$\|\alpha_k(US_i) - g_{k_i}US_i\| < \delta/2,
\|\Phi(US_i) - US_i\| = \|\Phi(U) \sum_j S_j S_i S_j^* - US_i\| 
\leq \|\Phi(U)V - U\| < \delta/2.$$

Let

$$T_1 = US_1, \ T_2 = US_2 US_1, \ T_3 = US_3^2.$$

Then $T_i$ generates a copy of $O_3$ in $O_2$ and satisfies

$$\|\alpha_k(T_i) - \mu_{ki}T_i\| < \delta,
\|\Phi(T_i) - T_i\| < \delta.$$
Thus one obtains unitaries $u_1, u_2$ in the C*-subalgebra generated by $T_i$ such that

$$
\|\alpha_k(u_i) - \frac{g_k}{u_i}\| < \epsilon/2,
\|\Phi(u_i) - u_i\| < \epsilon/2.
$$

Let, for $i = 1, 2$,

$$
S'_i = US_iu_i.
$$

Then

$$
\sum_i S'_i S'^*_i = 1,
\|\alpha_k(S'_i) - S'_i\| < \delta/2 + \epsilon/2 < \epsilon,
\|\Phi(S'_i) - S'_i\| < \delta/2 + \epsilon/2 < \epsilon.
$$

Hence $S'_i$ satisfies the desired properties.

### 3 Crossed products by $R$

Let $\alpha$ be a one-parameter automorphism group of the Cuntz algebra $O_n$ such that

$$
\alpha_t(S_k) = e^{i\lambda_k t} S_k, \ k = 1, \ldots, n
$$

where $\lambda_k$ are real constants. It is known [17] that $O_n \times_\alpha R$ is simple if and only if for all $i = 1, \ldots, n$ the closed subsemigroup of $R$ generated by $\lambda_1, \ldots, \lambda_n$ and $-\lambda_i$ is $R$. We have two cases in this situation: 1) All $\lambda_k$ are positive or negative and 2) the closed subsemigroup generated by all $\lambda_k$ is $R$. In the first case the system $(O_n, R, \alpha)$ has a unique KMS state at the inverse temperature $\beta$ defined by

$$
\sum_{k=1}^n e^{-\beta \lambda_k} = 1
$$

and hence $O_n \times_\alpha R$ is finite (cf. [9, 19]). In the second case we shall show:

**Theorem 2** Let $(O_n, R, \alpha)$ be as above and suppose that $\lambda_1, \ldots, \lambda_n$ generate $R$ as a closed subsemigroup. Then the crossed product $O_n \times_\alpha R$ is a stable, purely infinite, simple C*-algebra.

**Remark 5** In the case of $O_\infty$, a one-parameter unitary group $u$ on a separable infinite dimensional Hilbert space gives rise to a quasi-free action $\alpha$ on $O_\infty$ (cf. [9]) and $O_\infty \times_\alpha R$ is simple if and only if the spectrum of $u$ generates $R$ as a closed subsemigroup [17]. If $O_\infty \times_\alpha R$ is simple, then $\alpha$ has no KMS states for any inverse temperature $\beta$ and $O_\infty \times_\alpha R$ is infinite. The latter assertion follows from a modification of the proof of Lemma 6 below using [2, 1.2]. Moreover if the eigenvalues of $u$ generate $R$ as a closed subsemigroup, the proof of the above theorem can be extended immediately: $O_\infty \times R$ is stable and purely infinite.

A C*-algebra $A$ is called purely infinite if any non-zero hereditary C*-subalgebra contains an infinite projection and a projection is called infinite if it is equivalent to a proper subprojection in the sense of Murray and von Neumann (cf. [21]). We note that the stableness can be shown in a more general situation.

**Proposition 3** Let $A$ be a unital purely infinite simple C*-algebra and let $\alpha$ be a one-parameter automorphism group of $A$ such that $A \times_\alpha R$ is simple. Then $A \times_\alpha R$ is stable, i.e., $A \times_\alpha R$ is isomorphic to $(A \times_\alpha R) \otimes K$ where $K$ is the compact operators on a separable infinite dimensional Hilbert space.

**Remark 6** If $A$ is non-unital but separable, then the conclusion of the proposition still holds, as easily seen from its proof.

We note that $O_n \times_\alpha R$ cannot have both a non-zero projection and a non-zero trace, as seen from a general result:
Proposition 4 Let $A$ be a purely infinite simple $C^*$-algebra and let $\alpha$ be a one-parameter automorphism group of $A$ such that $A \times_\alpha R$ is simple. Then $A \times_\alpha R$ is either projectionless or traceless.

Proof. First note that we may assume that $A$ is unital. Let $p$ be a non-zero projection in the domain of the infinitesimal generator $\delta$ of $\alpha$ and define

$$\delta' = \delta - \text{ad } h$$

where $h = [\delta(p), p]$ and $\text{ad } h$ is given by $\text{ad } (x) = [h, x] = hx - xh$, $x \in A$; note that $\delta'(p) = 0$ and, hence, $p$ is $\alpha$-invariant where $\alpha'$ is the one-parameter automorphism group generated by $\delta'$. Observe that $A \times_\alpha R \cong A \times_{\alpha'} R$; hence, one has $pA \times_\alpha Rp \cong pA \times_{\alpha'} R$. Since $A \times_\alpha R$ contains a dense union of hereditary $C^*$-subalgebras of the form $pA \times_\alpha R$, it suffices to prove the conclusion for $pA \times_{\alpha'} R$.

We shall show that if $A$ is unital and $A \times_\alpha R$ has a non-zero projection and a non-zero densely defined lower semicontinuous trace, then $A$ must have a trace, which is a contradiction.

Let $P$ be the set of non-zero projections of $A \times_\alpha R$. Then $P$ is invariant under the automorphism group of $A \times_\alpha R$, in particular, under the dual action $\hat{\alpha}$ of $R = R$. Let $I$ be the linear span of $pA \times_\alpha Rq$, $p, q \in P$. Then $I$ is an $\hat{\alpha}$-invariant (dense) ideal: indeed, if $u$ is a unitary of $(A \times_\alpha R)^+ = A \times_\alpha R + C1$, then

$$pA \times_\alpha Rqu = pA \times_\alpha Ru^*qu \subset I$$

and any element of $A \times_\alpha R$ is a linear combination of unitaries of $(A \times_\alpha R)^+$. (I is the Pedersen ideal; see [23, 5.6].) Let $\tau$ be a non-zero densely defined lower semicontinuous trace on $A \times_\alpha R$. Then $P$ is in the domain of $\tau$ and $\tau \circ \hat{\alpha}_s(p) = \tau(p)$ for any $s \in R$ and $p \in P$. Note that for any $x \in I$, $s \rightarrow \tau \circ \hat{\alpha}_s(x)$ is bounded on $R$, since if $x = pxq$ then

$$|\tau \circ \hat{\alpha}_s(pxq)| \leq \|x\|\tau(p)^{1/2}\tau(q)^{1/2}.$$ 

Hence one can take an invariant mean of $\tau \circ \hat{\alpha}_s$ over $s \in R$. In this way we obtain an $\hat{\alpha}$-invariant trace $\tau_1$ defined on $I$, which is non-zero because $\tau_1(p) = \tau(p)$, $p \in P$. Then $\tau_1$ extends to a lower semicontinuous trace on $A \times_\alpha R$, which must be $\hat{\alpha}$-invariant, and by [19, 3.2] we obtain a trace on $A$.

Remark 7 In the situation of Proposition 4 further suppose that $A$ is nuclear (or exact). Then since $A \times_\alpha R$ is stable, simple, and nuclear [11, Prop. 14] (or exact [14, Prop. 7.1v]), it cannot be both projectionless and traceless [2, 12, 3, 16]. Hence, combining with Proposition 4, $A \times_\alpha R$ is either finite and projectionless or infinite (i.e., has infinite projections).

We now go to the proofs of Theorem 2 and Proposition 3 in this order.

Since $O_n \times_\alpha R$ is stable, simple, and nuclear, and is seen to have no traces (cf. [19]), one can conclude that it has an infinite projection. This fact is also shown by constructing a scaling element [2, 1.2].

Lemma 6 Under the situation of Theorem 2, $O_n \times_\alpha R$ has a scaling element.

Proof. Let $\lambda > 0$. There are finite sequences $\{m_i\}$, $\{k_i\}$ of non-negative integers such that

$$\lambda < \sum_{i=1}^n m_i \lambda_i = \lambda_V < 2\lambda,$$

$$-\lambda > \sum_{i=1}^n k_i \lambda_i = \lambda_U > -2\lambda.$$ 

Let

$$V = S_1^{m_1} \cdots S_n^{m_n}, U = S_n^{k_n} \cdots S_1^{k_1}.$$ 

Then $V$ and $U$ are isometries such that

$$U^*V = 0, \alpha_t(V) = e^{i\lambda_V t}V, \alpha_t(U) = e^{i\lambda_U t}U.$$
Let $g$ and $h$ be continuous functions on $\mathbf{R}$ such that

\[
0 \leq g \leq 1, \quad 0 \leq h \leq 1,
\]

\[
\text{supp} \, g \subset [-2\lambda_V - \delta, \delta], \quad \text{supp} \, h \subset [-\delta, -2\lambda_U + \delta],
\]

\[
g^2 + h^2 = 1 \text{ on } [-2\lambda_V, -2\lambda_U],
\]

where $\delta > 0$ is sufficiently small. Let

\[
x = Vg(H) + Uh(H) \in O_n \times_\alpha \mathbf{R}
\]

where $H$ is the generator of the canonical one-parameter unitary group in the multiplier algebra $M(O_n \times_\alpha \mathbf{R})$ which implements $\alpha$. Then

\[
x^*x = g^2(H) + h^2(H),
\]

\[
xx^* = g^2(H - \lambda_V)VV^* + h^2(H - \lambda_U)UU^* + gh(H - \lambda_V)UV^* + gh(H - \lambda_U)UV^*.
\]

Since $\text{supp} \, g(\cdot - \lambda_V) \subset [-\lambda_V - \delta, \lambda_V + \delta]$ and $\text{supp} \, h(\cdot - \lambda_U) \subset [\lambda_U - \delta, -\lambda_U + \delta]$, it follows that $x$ is a scaling element:

\[
x^*x \cdot xx^* = xx^*, \quad x^*x \neq xx^*.
\]

**Lemma 7** Let $A$ be a $C^*$-algebra and $B$ a hereditary $C^*$-subalgebra of $A$. Let $\{p_k\}$ be a sequence in $A$ such that $0 \leq p_k \leq 1$, $p_k p_{k+1} = p_{k+1}$, and the limit of $p_k$ in $A^{**}$ is a minimal projection and belongs to $B^{**}(\subset A^{**})$. Then there is a sequence $\{b_k\}$ in $B$ such that $0 \leq b_k \leq 1$ and $\|p_k - b_k\| \to 0$.

**Proof.** There is a unique pure state $\phi$ on $A$ such that $\phi(p_k) = 1$ for all $k$; moreover, one has $\|\phi|B\| = 1$. There are $a, b \in B$ such that $0 \leq a \leq 1$, $0 \leq b \leq 1$, $ab = a$, and $\phi(a) = 1$. Let $E$ be the support projection of $a$ in $B^{**} \subset A^{**}$:

\[
E = \lim_{k \to \infty} a^{1/k}.
\]

We assert that $\|p_k(1 - E)\| \to 0$ as $k \to \infty$. Here we note that $\{|\|p_k(1 - E)\|\}$ is non-increasing. Suppose that $\|(1 - E)p_k(1 - E)\| > \delta$ for all $k$. Then there are states $\psi_k$ such that $\psi_k(E) = 0$, $\psi_k(p_k) > \delta$. Let $\psi$ be a weak* limit point of $\psi_k$. Then since $\psi_k(a) = 0$ and $\psi(p_k) > \delta$ for $l \geq k$, one obtains that

\[
\psi(a) = 0, \quad \psi(p_k) \geq \delta.
\]

Let $p = \lim p_k$ in $A^{**}$. Then $\psi(p) \geq \delta$. Since $p$ is a minimal projection of $A^{**}$, and $pa = ap = p$, it follows that

\[
\psi(a) = \psi(pa + (1 - p)a) \geq \psi(pa) \geq \delta
\]

which is a contradiction. Hence, since $E \leq b \leq 1$, it follows that $\|p_k(1 - b)\| \to 0$ as $k \to \infty$. Let $b_k = bp_k b$. Then $b_k \in B$ and $\|b_k - p_k\| = \|(b - 1)p_k b + p_k(b - 1)\| \to 0$.

**Proof of Theorem 2** Let $\mu$ be an aperiodic sequence of $\{1, 2, \ldots, n\}$, i.e., $\mu \in \{1, \ldots, n\}^\mathbb{N}$ and for any $k \neq l$, $(\mu_k, \mu_{k+1}, \ldots)$ is different from $(\mu_l, \mu_{l+1}, \ldots)$. Let $\phi$ be a state of $O_n$ such that

\[
\phi(S_{\mu(k)} S_{\mu(k)}^*) = 1, \quad k \in \mathbb{N}
\]

where

\[
\mu(k) = (\mu_1, \ldots, \mu_k), \quad S_{\mu(k)} = S_{\mu_1} \cdots S_{\mu_k}.
\]

Then $\phi$ is a unique pure state [7]. Since $\phi$ is $\alpha$-invariant, $\phi$ extends to a pure state $\overline{\phi}$ of $O_n \times \mathbf{R}$ with

\[
\overline{\phi}(u_t) = 1, \quad t \in \mathbf{R}.
\]

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(More precisely, the unique extension of $\tilde{\phi}$ to $M(O_n \times \mathbb{R})$ restricts to $\phi$ on $O_n$.) Let $f_k$ be a continuous function on $\mathbb{R}$ such that

$$f_k(t) = \begin{cases} 1 & |t| \leq \frac{1}{k+1} \\ 0 & |t| \geq \frac{1}{k} \end{cases}$$

and is linear elsewhere, and let $e_k = f_k(H)$ where $u_t = e^{itH}$. Then $p_k = S_{\alpha(k)}S_{\alpha(k)}^* e_k$ forms a decreasing sequence of positive elements of $O_n \times \mathbb{R}$ such that $p_k p_{k+1} = p_{k+1}$ and the limit in the second dual is the support projection of $\tilde{\phi}$. Let $B$ be a non-zero hereditary C*-subalgebra of $O_n \times \mathbb{R}$. Then by Kadison’s transitivity theorem there is a unitary $u \in (O_n \times \mathbb{R})^+ = O_n \times \mathbb{R} + C1$ such that

$$\|\overline{\phi}|Bu^*\| = 1.$$  

To find an infinite projection in $B$ it suffices to find one in the hereditary C*-subalgebra $Bu^*$. Hence we now assume that $\|\overline{\phi}|B\| = 1$. Fix $k$ and let $E$ be an infinite projection of $O_n \times \mathbb{R}$. Since $O_n \times \mathbb{R}$ is simple, there are $x_1, \ldots, x_m \in O_n \times \mathbb{R}$ such that

$$\sum_{i=1}^m x_i p_k x_i^* = E.$$  

One finds $m$ isometries $V_1, \ldots, V_m \in O_n$ of the form $S_{\mu(k)}S_{\mu}$ with $a$ a finite sequence such that

$$\alpha_t(V_j) = e^{i\lambda_{V_j}t}V_j$$

where $\lambda_{V_j}$ is so close to 0 that $V_j$ almost commutes with $e_k$, and $\{V_j V_j^* ; j = 1, \ldots, m\}$ are mutually orthogonal. Define

$$x = \sum_{i=1}^m x_i S_{\mu(k)}S_{\mu(k)}^* V_i^* e_k^1/k.$$  

Then

$$xx^* \approx \sum_{i,j} x_i S_{\mu(k)}S_{\mu(k)}^* V_i^* V_j^* e_k S_{\mu(k)}S_{\mu(k)}^* x_j$$

$$= \sum_{i} x_i e_k S_{\mu(k)}S_{\mu(k)}^* x_i$$

$$= E,$$

where $\approx$ means *approximately equal* (which can be made arbitrarily precise depending on the choice of elements involved). Hence $x^* x$ is close to a projection and $x = xp_{k-1}$. By Lemma 7, for a sufficiently large $k$, there is a $b \in B$ such that $0 \leq b \leq 1$ and $p_{k-1} \approx b$. Then $xb \approx x$ and $bx^*xb \in B$ is close to a projection. Thus by functional calculus, one obtains a projection in $B$ which is equivalent to $E$. This concludes the proof that $O_n \times \mathbb{R}$ is purely infinite. Since $O_n \times \alpha \mathbb{R}$ is separable and does not have an identity, it is automatically stable.

**Lemma 8** Let $A$ be a C*-algebra and let $\{p_k\}$ be a sequence of projections of $A$ such that

$$\|p_k x - x\| \to 0$$

for any $x$ in the C*-subalgebra generated by $p_1, p_2, \ldots$. Then there are a subsequence $\{n_k\}$ of positive integers and an increasing sequence $\{e_k\}$ of projections of $A$ such that $\|p_{n_k} - e_k\| \to 0$.

In the proof we use the following lemma whose proof is standard:

**Lemma 9** There exists a constant $c > 0$ satisfying the following condition: For any sufficiently small $\epsilon > 0$ and any projections $e, p \in A$ with $\|ep - p\| < \epsilon$, there is a unitary $u$ in the C*-subalgebra generated by $1, e, p$ such that

$$upu^* \leq e, \quad \|u - 1\| < c\epsilon.$$
Proof of Lemma 8 We construct a subsequence \( n_1 = 1, n_2, \ldots \) of positive integers and a sequence \( u_1 = 1, u_2, \ldots \) of unitaries such that

\[
\begin{align*}
&u_k \in C^*(1, p_{n_1}, \ldots, p_{n_k}), \\
\{u_k p_{n_k} u_k^*\} \text{ is increasing}, \\
&\|1 - u_k\| < 1/k. 
\end{align*}
\]

Suppose we have constructed these up to \( k \). Then noting that \( u_k p_{n_k} u_k^* \in C^*(p_{n_1}, \ldots, p_{n_k}) \) choose \( n_{k+1} > n_k \) such that

\[
\|p_{n_{k+1}} u_{k+1} u_{k+1}^* - u_k p_{n_k} u_k^*\| < \frac{1}{c(k+1)}.
\]

Then there exists a unitary \( u_{k+1} \in C^*(1, p_{n_1}, \ldots, p_{n_{k+1}}) \) such that

\[
\begin{align*}
&u_{k+1} p_{n_{k+1}} u_{k+1}^* \geq u_k p_{n_k} u_k^*, \\
&\|1 - u_{k+1}\| \leq 1/(k+1). 
\end{align*}
\]

Thus \( e_k = u_k p_{n_k} u_k^* \) and \( n_k \) satisfy the desired properties.

Proof of Proposition 3 It suffices to construct a family of matrix units \( \{e_{ij}; i, j = 1, 2, \ldots\} \) in the multiplier algebra \( M(A \times R) \) such that

\[
\sum_{i=1}^{\infty} e_{ii} = 1
\]

in the strict topology. We construct such \( \{e_{ij}\} \) in the \( C^* \)-subalgebra

\[
B = \bigcup_{k=1}^{\infty} A \times \alpha_{3^{-k}} Z
\]

of \( M(A \times R) \). Since \( A \times \alpha_{3^{-k}} Z \) is purely infinite by [13] or Lemma 10 below, \( B \) is also purely infinite. Define continuous functions \( \phi_k \) on \( R \) as follows:

\[
\phi_k(t) = \begin{cases} 
0 & t \leq -b_k - 1 \\
-\frac{b_k}{b_k - 1} & -b_k - 1 \leq t \leq -b_k \\
1 & -b_k \leq t \leq b_k \\
-\frac{b_k + 1}{b_k} & b_k \leq t \leq b_k + 1 \\
0 & b_k \leq t 
\end{cases}
\]

where

\[
b_k = \sum_{j=1}^{k} 3^j = \frac{3^k - 1}{2}.
\]

Define

\[
f_k(t) = \sum_{l=-\infty}^{\infty} \phi_k(2\pi(t + 3^{k+1}l)).
\]

Then \( \{f_k(H)\} \) is an increasing sequence in \( B \), and

\[
\bigcup_k f_k(H)(A \times R) f_k(H)
\]

is dense in \( A \times R \). Then one can choose projections \( p_k \in f_k(H)B f_k(H) \) such that

\[
\|p_k f_{k-1}(H) - f_{k-1}(H)\| < 1/k.
\]

We may further assume that \( [p_k] = 0 \in K_0(B) \) by replacing it by a bigger projection. Then it follows that for any \( x \in A \times R \) and \( x \) in the \( C^* \)-subalgebra generated by \( p_1, p_2, \ldots \), one has that

\[
\|p_k x - x\| \to 0.
\]
Then by Lemma 8 one can construct a strictly increasing sequence \( \{e_k\} \) of projections of \( B \) such that \( [e_k] = 0 \), and
\[
\|e_k x - x\| \to 0, \ x \in A \times \mathbb{R}.
\]
Then
\[
e_1 + \sum_{k=1}^{\infty} (e_{k+1} - e_k) = 1
\]
in the strict topology. Since the projections \( e_1, e_{k+1} - e_k \) with \( k = 1, 2, \ldots \) are all equivalent (cf. [8]), one can complete them to matrix units.

**Lemma 10** Let \( A \) be a purely infinite simple \( C^* \)-algebra, \( G \) a discrete group, and \( \alpha \) an action of \( G \) on \( A \) such that for each \( g \in G \setminus \{e\} \), \( \alpha_g \) is outer. Then the reduced crossed product \( A \times_{\alpha} G \) is a purely infinite simple \( C^* \)-algebra.

**Proof.** It is shown in [18] that the reduced crossed product is simple. The same proof works for showing that it is purely infinite as follows.

Let \( a \) be a positive element of \( A \times_{\alpha} G \) with \( \|a\| = 1 \) and let \( E \) be an infinite projection of \( A \subset A \times G \).

We shall show that there is a \( z \in A \times G \) such that \( zaE = E \).

Let \( u_g, \ g \in G \) be the standard unitaries in the multiplier algebra of \( A \times_{\alpha} G \) implementing \( \alpha \). We approximate \( a^{1/2} \) by an element \( b \) of the dense *-algebra spanned by \( Au_g, \ g \in G \), and hence \( a \) by \( c = b^* b = \sum_{i=0}^{n} c_i u_g \), where \( c_i \in A, \ g_0 = e \), the identity of \( G \), and \( g_0, \ldots, g_n \) are distinct elements of \( G \). Let \( a_0 \) be the coefficient of \( u_e = 1 \) for \( a \). Then \( a_0 \) is a non-zero positive element of \( A \) as well as \( c_0 \) and \( \|a_0 - c_0\| \leq \|a - c\| \).

By 3.2 of [18], for any \( \epsilon > 0 \) one finds a positive \( x \in A \) such that \( \|x\| = 1 \),
\[
\|xc_0 x\| > (1 - \epsilon)\|c_0\|, \\
\|xc_i u_g x\| < \epsilon/n, \ i = 1, \ldots, n.
\]

Define continuous functions \( f \) and \( g \) on \( \mathbb{R} \) by \( f(t) = \max(0, t - (1 - \epsilon)\|c_0\|) \) and \( g(t) = \min(t, (1 - \epsilon)\|c_0\|) \).

For \( d = f(xc_0 x) \) one finds a \( y_1 \in A \) such that \( y_1 d y_1^* = E \). Then \( y_2 = (1 - \epsilon)^{-1/2}\|c_0\|^{-1/2} y_1 d^{1/2} \) has norm \( (1 - \epsilon)^{-1/2}\|c_0\|^{-1/2} \) and satisfies that \( y_2 p y_2^* = E \) where \( p = g(xc_0 x) \). Since \( y_2 xc_0 x y_2^* \geq y_2 p y_2^* = E \), one obtains a \( y \in A \) such that
\[
yxc_0 xy^* = E, \\
\|y\| \leq (1 - \epsilon)^{-1/2}\|c_0\|^{-1/2}.
\]

Since \( \|yx xy^* - E\| \leq \|yx xy^* - E\| + \|yx c_i u_g yx^*\| < \epsilon(1 - \epsilon)^{-1}\|c_0\|^{-1} \), it follows that
\[
\|yx ax y^* - E\| \leq (1 - \epsilon)^{-1}\|c_0\|^{-1}(\|a - c\| + \epsilon).
\]

If \( \|a - c\| \) and \( \epsilon \) are sufficiently small, then there is a \( z \in A \times G \) such that \( xyz ax yz^* = E \). This completes the proof.

**Remark 8** The proof of Lemma 10 can be modified to prove the following: In the situation of Lemma 10 with \( A \) not necessarily purely infinite, if \( A \) has (SP) then \( A \times_{\alpha} G \) has (SP), where (SP) is the property that every non-zero hereditary \( C^* \)-subalgebra contains a non-zero projection. See, e.g., [1] for (SP) and other related properties.

**References**


[12] U. Haagerup, Quasitraces on exact $C^*$-algebras are traces, preprint.


