Fell bundles associated to groupoid morphisms

Valentin Deaconu, Alex Kumjian and Birant Ramazan
University of Nevada, Reno

Groupoids in operator algebras and noncommutative geometry,
Institut Henri Poincaré, Paris, 27 February 2007

Given a continuous open surjective morphism $\pi : G \to H$ of étale groupoids with amenable kernel, we construct a Fell bundle E over H and prove that its C*-algebra $C_r^*(E)$ is isomorphic to $C_r^*(G)$. The case $H = X$, a locally compact space, was treated earlier by Ramazan. We conclude that $C_r^*(G)$ is strongly Morita equivalent to a crossed product, the C*-algebra of a Fell bundle arising from an action of the groupoid H on a C*-bundle over H^0. We prove a structure theorem for abelian Fell bundles.

Introduction

Fell introduced the notion of C*-algebraic bundle over a group in 1977 (a modification of his notion of Banach *-algebraic bundles) as a tool to study induced representations of locally compact groups.

Yamagami introduced the natural generalization of this concept to groupoids in 1991 (the object was called a C*-algebra over a groupoid) and proved a Gootman-Rosenberg type theorem for the primitive ideals of the associated C*-algebra.

This notion was further studied under the name Fell bundle by Muhly, Kumjian and others. Roughly speaking a Fell bundle restricts to a (continuous) C*-bundle over the unit space and (under mild hypotheses) every other fiber is a strong Morita equivalence bimodule between the appropriate fibers over the unit space. The notion generalizes that of an action of a groupoid on a C*-bundle fibered over the unit space.

Objective

The goal of this work is to prove an analog of a theorem of Ramazan. Slightly reformulated the result asserts that if G is a locally compact amenable groupoid, X is a locally compact space and $\pi : G \to X$ is a continuous open surjection such that π is a groupoid homomorphism when X is viewed as a trivial groupoid, then there is a a C*-bundle F over X with $F_x = C^*(\pi^{-1}(x))$ such that $C_0(F) \cong C^*(G)$.

The result was proved by Lee when G is also required to be a trivial groupoid.

We prove the analogous result where X is replaced by a groupoid and both groupoids are required to be étale.

We also prove a structure theorem for abelian Fell bundles. A Fell bundle E is said to be abelian if E_u is abelian for every $u \in G^0$.

Definition

Let G be a locally compact Hausdorff groupoid with unit space G^0 with range and source maps r and s. A Banach bundle $p : E \to G$ is said to be a *Fell bundle* if there is a continuous bilinear multiplication map $E_{g_1} \times E_{g_2} \to E_{g_1 g_2}$ (write $(e_1, e_2) \mapsto e_1 e_2$) for all $g_1, g_2) \in G^2$ and a conjugate linear involution $E_g \to E_{g^{-1}}$ (write $e \mapsto e^*$) satisfying the following conditions.

i. $(e_1 e_2)e_3 = e_1(e_2 e_3)$ whenever defined

ii. $||e_1 e_2|| \leq ||e_1|| \, ||e_2||$

iii. $(e_1 e_2)^* = e_2^* e_1^*$

iv. $||e^* e|| = ||e||^2$

v. $e^* e \geq 0$

We say E is *saturated*, if the range of each multiplication map is total, and *abelian* if E_u is abelian for all $u \in G^0$.

Fell line bundles

Let Γ be a proper $\mathbb{T}$-groupoid over H, that is, Γ is a groupoid endowed with the structure of a principal $\mathbb{T}$-bundle over H compatible with groupoid structure. Form the associated line bundle:

$$E = \Gamma *_{\mathbb{T}} \mathbb{C} = (\Gamma \times \mathbb{C})/\mathbb{T}$$

(where $t(\gamma, z) = (t\gamma, t^{-1}z)$). One defines multiplication and involution as follows:

$$(\gamma_1, z_1)(\gamma_2, z_2) = (\gamma_1\gamma_2, z_1z_2),$$
$$(\sigma, z)^* = (\sigma^{-1}, \overline{z}).$$

One verifies that E is a Fell bundle over H with these operations and that $E_u \cong \mathbb{C}$ for all $u \in H^0$; so E is an abelian Fell bundle. A bundle of this type is called a *Fell line bundle*. Any Fell bundle for which E_h is one dimensional for all $h \in H$ is of this type.

Basic facts

Given a Fell bundle E over a groupoid G, note that E_u is a C*-algebra for every $u \in G^0$. Moreover the restriction $E^0 = E|_{G^0}$ is a C*- bundle and so $C_0(E^0)$ is a C*-algebra.

For each $g \in G$, E_g may be viewed as a right Hilbert $E_{s(g)}$-module or a left Hilbert $E_{r(g)}$-module (when endowed with the natural inner-products; moreover, if E is saturated, E_g is an $E_{r(g)}$–$E_{s(g)}$ equivalence bimodule.

We define a Hilbert module $L^2(E)$ over $C_0(E^0)$ as the completion of $C_c(E)$ in the appropriate norm. The $*$-algebra $C_c(E)$ acts on $L^2(E)$ in a natural way and so allows us to define the reduced C*-algebra $C_r^*(E)$.

The reduced C*-algebra of a Fell bundle

For G an étale groupoid and $p : E \to G$ a Fell bundle, we define multiplication and involution on the space of compactly supported continuous sections $C_c(E)$; for $\xi, \eta \in C_c(E)$ and $g \in G$

$$(\xi\eta)(g) = \sum_{g=g_1 g_2} \xi(g_1)\eta(g_2), \quad \xi^*(g) = \xi(g^{-1})^*.$$

Define an inner product $\langle \xi, \eta \rangle = P(\xi^*\eta)$ for $\xi, \eta \in C_c(E)$, where $P : C_c(E) \to C_c(E^0)$ is the restriction map; the completion $L^2(E)$ is a right Hilbert module over $C_0(E^0)$ (with norm defined by this inner product $\|\xi\|^2 = \|\langle \xi, \xi \rangle\|$). Observe that $C_c(E)$ acts by left multiplication on $L^2(E)$.

The C*-algebra $C_r^*(E)$ is defined as the completion of $C_c(E)$ in $\mathcal{L}(L^2(E))$ (in the operator norm).

Main Result

Lemma (Ramazan): Let K be an amenable groupoid and X a space. Given a continuous open morphism $\pi : K \to X$, there is a C*-bundle F over X with fibers $C_r^*(K(x))$, where $K(x) = \pi^{-1}(x)$. Moreover, $C_r^*(K) \cong C_0(F)$.

This is the key ingredient in proving the following:

Theorem: Given an open surjective morphism of étale groupoids $\pi : G \to H$ with amenable kernel $K := \pi^{-1}(H^0)$, there is a Fell bundle $E = E(\pi)$ over H such that $C_r^*(G) \cong C_r^*(E)$. Moreover, E is saturated.

Sketch of proof:
Set $E_h = \overline{C_c(\pi^{-1}(h))}^{||\cdot||}$ for an appropriate choice of norm.

Fell's examples

We first discuss a class of examples considered by Fell. Let G be a discrete group, let K be a normal subgroup and let $H = G/K$ with $\pi : G \to H$ the canonical morphism. Then we get a Fell bundle E over H with the fiber $C_r^*(K)$ over the identity element, such that $C_r^*(G) \cong C_r^*(E)$. It is equivalent to think of this structure as a grading of $C_r^*(G)$ by H. Since the groups are discrete the bundle "transverse" bundle structure is trivial.

If we specialize to the case where G is the integer Heisenberg group with $K \cong \mathbb{Z}$ its center, the $G/K \cong \mathbb{Z}^2$ grading arises from the well-known action of the torus $\mathbb{T}^2$ on $C^*(G)$. The fibers E_h are all isomorphic to $C(\mathbb{T})$.

Graphs

Let V, W be finite graphs. A graph morphism $\phi : V \to W$ is a map which preserves incidences. If ϕ is surjective, we say that it has the path lifting property if for any $v \in V^0$ and $b \in W^1$ such that $s(b) = \phi(v)$, there is $a \in V^1$ such that $s(a) = v$ and $\phi(a) = b$.

Recall that $C^*(V)$ is the C*-algebra of the amenable groupoid

$$G_V = \{(x, p-q, y) \in X_V \times \mathbb{Z} \times X_V \mid \sigma^p(x) = \sigma^q(y)\},$$

where σ is the shift map and X_V is the space of infinite paths. With ϕ as above we obtain a surjective open morphism $\pi : G_V \to G_W$ given by $\pi(x, k, y) = (\tilde{\phi}(x), k, \tilde{\phi}(y))$ between the associated groupoids, with kernel

$$K = \{(x, 0, y) \in G_V \mid \tilde{\phi}(x) = \tilde{\phi}(y)\}.$$

Hence, our theorem applies.

Groupoid actions

Definition: Let G be a groupoid, let X be a space and let $\rho : X \to G^0$ be a continuous open surjection. Then an action of G on X is given by a continuous map $\alpha : G * X \to X$, where $G * X = \{(g, x) : s(g) = \rho(x)\}$, write $\alpha(g, x) = g \cdot x$, such that

i. For all $(g, x) \in G * X$, $\rho(g \cdot x) = r(g)$

ii. For all $x \in X$, $\rho(x) \cdot x = x$

iii. For $(g_1, g_2) \in G^2$, $(g_2, x) \in G * X$, we have

$$(g_1 g_2, x), (g_1, g_2 \cdot x) \in G * X \quad \text{and} \quad g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

Given such an an action, the *transformation groupoid* $G \ltimes X$ is constructed from $G * X$ with unit space X such that

$$r(g, x) = g \cdot x, \quad s(g, x) = x, \quad (g_1 g_2, x) = (g_1, g_2 \cdot x)(g_2, x).$$

Note that the projection map $G \ltimes X \to G$ is a morphism.

Groupoid coverings

Let $\pi : G \to H$ be a open surjective morphism of groupoids. Then π is called a *groupoid covering* if for any $h \in H$ and $x \in G^0$ with $\pi(x) = s(h)$, there is a unique $g \in G$ with $s(g) = x$ and $\pi(g) = h$.

Note that $\pi^{-1}(H^0) = G^0$.

Example: Suppose that a groupoid H acts on the space X. Then the projection map $H \ltimes X \to H$ is a covering.

Suppose that $\pi : G \to H$ is a groupoid covering where G and H are étale groupoids and let $E = E(\pi)$. Then $E_u = C_0(\pi^{-1}(u))$ for all $u \in H^0$ and therefore E is an abelian Fell bundle.

Abelian extensions of groupoids

Let $\pi : G \to H$ be a open surjective morphism where G and H are étale groupoids. Suppose that the restriction of π to G^0 induces a homeomorphism $G^0 \cong H^0$ and that $A = \pi^{-1}(H^0)$ is a sheaf of abelian groups over G^0. Then the resulting sequence:

$$A \xrightarrow{\iota} G \xrightarrow{\pi} H$$

is called an abelian extension where ι is the inclusion map.

Since A is amenable, the main theorem applies and we get a Fell bundle $E = E(\pi)$. Moreover, $E_u = C^*(A_u)$ for all $u \in H^0$, where A_u is the fiber of A over u. Since A is a sheaf of abelian groups, E_u is abelian for all $u \in H^0$. Hence, E is an abelian Fell bundle.

Structure Theorem

In the class we consider, every abelian Fell bundle arises as some combination of the above examples:

Theorem: Given a saturated abelian Fell bundle E over an étale groupoid H, there is an étale groupoid G, a groupoid covering $\pi : G \to H$ and a Fell line bundle L over G such that

$$E_u \cong C_0(\pi^{-1}(u)) \quad \text{for all } u \in H^0$$

and $C_r^*(L) \cong C_r^*(E)$.

Sketch of proof: Since E is an abelian Fell bundle, $C_0(E^0)$ is an abelian C*-algebra. Set $X = \widehat{C_0(E^0)}$ i.e. $C_0(E^0) = C_0(X)$ and observe that since $C_0(X)$ is the C*-algebra of a bundle over H^0, we get a continuous open surjection $\rho : X \to H^0$. For each $h \in H$, E_h is a $E_{r(h)}$-$E_{s(h)}$ equivalence bimodule and hence we get a homeomorphism $\alpha_h : X_{s(h)} \to X_{r(h)}$ (note $E_u \cong C_0(X_u)$).

Key lemmas

Lemma A: Let E be a saturated Fell bundle over an étale groupoid G and let U be an open G-set. Then the completion of $C_c(U, E)$, with respect to the the supremum norm is an A-B equivalence bimodule when endowed with the natural inner products and actions, where $A = C_0(r(U), E)$ and $B = C_0(s(U), E)$.

Lemma B (Raeburn): Let $\mathfrak{M}$ be an A-A equivalence bimodule, where $A \cong C_0(T)$ for some locally compact Hausdorff space T. Then there is a Hermitian line bundle M over T such that $\mathfrak{M} \cong C_0(M)$.

Rest of the sketch

By Lemma A the action $\alpha : H * X \to X$ is continuous because the homeomorphism $\alpha_h : X_{s(h)} \to X_{r(h)}$ extends to a homeomorphism $\rho^{-1}(s(U)) \to \rho^{-1}(r(U))$ where U is any open H-set containing h (note $X_u = \rho^{-1}(u)$). Thus we may set $G = H \ltimes X$ and define $\pi : G \to H$ to be the projection map.

We construct the Fell line bundle over G using Lemma B. Let U be an open H-set then $C_0(\rho^{-1}(r(U)))$ and $C_0(\rho^{-1}(s(U)))$ are abelian and Morita equivalent and therefore isomorphic. By the lemma the equivalence bimodule may be identified with the space of continuous sections vanishing at infinity of a Hermitian line bundle over the common spectrum which may be identified with $\pi^{-1}(U)$. Patching these line bundles together yields a Fell line bundle L over G so that $C_r^*(L) \cong C_r^*(E)$.

Selected References

Fell, *Induced representations and Banach *-algebraic bundles*, '77

Kumjian, *Fell bundles over groupoids*, '98

Lee, *On the C^*-algebras of operator fields*, '76

Muhly, Renault and Williams, *Continuous-trace groupoid C^*-algebras. III*, '96

Muhly, *Bundles over groupoids*, '01

Raeburn, *On the Picard group of a continuous trace C*-algebra*, '81

Ramazan, *Quantification par Déformation des Variétés de Lie-Poisson*, Ph.D. Thesis, '98

Yamagami, *On primitive ideal spaces of C*-algebras over certain locally compact groupoids,* "91