An involutive automorphism of the Bunce-Deddens algebra
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Very recently Blackadar has shown the existence of a period two automorphism of the CAR algebra for which the fixed point algebra is not AF [B2]. Motivated by his example we show that the crossed-product of the Bunce-Deddens algebra of type $2^\infty$ by the involutive automorphism which acts on the unit circle by complex conjugation is AF (the K-theory of this crossed-product may be found in [B1, 10.11.5c]). That this was likely to be true was suggested in personal correspondence from Blackadar. The methods used in the proof can be modified to verify the analogous result for any Bunce-Deddens algebra, but in the interests of brevity and ease of exposition we restrict ourselves to the case mentioned.

Let $T = R/Z$ and $D$ denote the dyadic rationals mod $1$; we view $D$ as a subgroup of $T$ in the obvious way. Thus, $D$ acts on $C(T)$ by translation and the resulting crossed-product $C(T) \times_\sigma D$, where $\tau_d f(x) = f(x-d)$, is the well-known Bunce-Deddens algebra of type $2^\infty$ (see [BD],[G]). For $f \in C(T)$ set $\sigma f(x) = f(-x)$; clearly, $\sigma^2 = 1$ and $\sigma \tau_d \sigma = \tau_{-d}$. We show below that the resultant crossed-product, $C(T) \times_\sigma D \times_\sigma Z_2$, is AF by showing that elements of this algebra may be approximated by elements of certain finite-dimensional subalgebras in a controlled way (see [Br, 2.2]). The heart of the proof lies in the construction of these subalgebras.

We construct a finite-dimensional subalgebra, $A_n$, of the algebra, $C(T) \times_\sigma Z_2^n \times_\sigma Z_2$, generated by $C(T)$ and unitaries, $u,v$, where $u$ is the unitary implementing translation by $2^{-n}$ and $v$ is the unitary implementing $\sigma$. Fix $\delta$ with $0 < \delta < 1/2^{n+1}$ and set $\theta_k = (2k+1)/2^{n+1}$. Note that $\theta_k$ is a fixed point of the homeomorphism of $T$ associated to the automorphism $\Ad u^{2^{k+1} v}$.

For $x \in [\theta_k - \delta, \theta_k + \delta]$ set $\varphi_k(x) = (\theta_k + \delta - x)/2\delta$ and define $f_k \in C(T)$ by

$$f_k(x) = \begin{cases} 1 - \varphi_k^{-1}(x) & \text{if } \theta_k - 1 - \delta \leq x < \theta_k - \delta, \\ 1 & \text{if } \theta_k - 1 + \delta < x < \theta_k - \delta, \\ \varphi_k(x) & \text{if } \theta_k - \delta \leq x < \theta_k + \delta, \\ 0 & \text{elsewhere}; \end{cases}$$

and define $g_k \in C(T)$ by

$$g_k(x) = \begin{cases} (\varphi_k(x)(1 - \varphi_k(x)))^{1/2} & \text{if } \theta_k - \delta \leq x \leq \theta_k + \delta, \\ 0 & \text{elsewhere}. \end{cases}$$

Finally, set

$$p_k = f_k + (-1)^k(u^{2^{k+1} v} g_k + u^{2^{k-1} v} g_{k-1}).$$

One verifies that $p_k$ is a projection for $k = 0, \ldots, 2^n - 1$ (see [R, 1.1]) and that

$$\sum_{k=0}^{2^n-1} p_k = 1.$$ 

Moreover,

$$v p_k v = p_{-k},$$

$$u^{2^j} p_k u^{-2^j} = p_{k-2^j}.$$  

The subalgebra generated by the projections, $p_k$, and the unitaries, $u^2$ and $v$, is easily seen to be finite dimensional (in fact, it is isomorphic to the direct sum of four copies of $M_{2^{n-1}}(C)$); denote this subalgebra by $A_n$.

To show that $C(T) \times_\sigma D \times_\sigma Z_2$ is AF, it suffices to show that any diagonal element, $f \in C(T)$, may be approximated by an element in $A_n$ (for $n$ sufficiently large). This is because arbitrary elements may be approximated by elements of the subalgebras, $C(T) \times_\sigma Z_2^n \times_\sigma Z_2$, which in turn may be written as finite sums of terms of the form, $w h$, where $w$ is a unitary (in $A_n$ for $n > m$) and $h \in C(T)$. Suppose that $n$ has
been chosen so that $|f(x) - f(y)| \leq \epsilon$ when $|x - y| \leq 2^{-n}$. Set $\lambda_k = f(k \frac{\pi}{2})$; the reader may wish to check that

$$\left\| f - \sum_{k=0}^{2^n-1} \lambda_k p_k \right\| \leq 2\epsilon.$$

References


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