Hausdorff Measures and KMS States

Alexander Kumjian

University of Nevada, Reno

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Based on work with Jean Renault and Marius Ionescu

Introduction.

Given a compact metric space (X, ρ) with Hausdorff dimension $\beta > 0$, a local homeomorphism $T : X \to X$ and a continuous real-valued function $\varphi : X \to \mathbb{R}$ such that for all $x \in X$:

$$e^{\varphi(x)} = \lim_{y \to x} \frac{\rho(Tx, Ty)}{\rho(x, y)},$$

then the Hausdorff measure defines an (α, β)-KMS state on $C^*(\Gamma)$ where α is the action of $\mathbb{R}$ defined by φ and Γ is the Renault-Deaconu groupoid.

Under mild assumptions the KMS state is unique.

This talk is based on joint work with Jean Renault (see [KR]) and ongoing joint work with Marius Ionescu.

Hausdorff Measure and Dimension.

Let (X, ρ) be a metric space. Fix $s > 0$; for $F \subset X$ and $\varepsilon > 0$, set

$$\overline{\mu}^s_\varepsilon(F) = \inf_{\mathcal{A} \in \mathfrak{A}_\varepsilon} \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^s$$

where $\mathfrak{A}_\varepsilon$ is the collection of countable ε-covers of F. Set

$$\overline{\mu}^s(F) = \lim_{\varepsilon \downarrow 0} \overline{\mu}^s_\varepsilon(F).$$

Note $\overline{\mu}^s$ is a metric outer measure and so defines a Borel measure μ^s on X.

Fact: For $F \subset X$ there is $s_0 \geq 0$, the *Hausdorff dimension* of F, denoted $\dim F$, such that $\overline{\mu}^s(F) = \infty$ if $s < s_0$ and $\overline{\mu}^s(F) = 0$ if $s > s_0$.

For $s = \dim X$, μ^s is called the *Hausdorff measure*.

We assume throughout that $0 < s < \infty$ and $0 < \mu^s(X) < \infty$.

KMS States.

In the C*-algebraic formulation of quantum statistical mechanics, KMS states are equilibrium states associated to the one-parameter group of automorphisms of time evolution (see [BR, 5.3]).

Definition

Let A be a C*-algebra, $\beta \in \mathbb{R}$ and let $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ be a strongly continuous action.

For $\beta \neq 0$, a state ω on A is said to be an (α, β)-*KMS state* if

$$\omega(b\alpha_{i\beta}(a)) = \omega(ab)$$

for all $a, b \in A$ with a entire for α.

A KMS state for $\beta = 0$ is an α-invariant tracial state.

The parameter β is called the *inverse temperature*.

Cantor set.

Fix $n \in \mathbb{N}$ with $n > 1$ and $0 < r_1, \ldots, r_n < 1$ and set

$$X := \{1, \ldots, n\}^{\mathbb{N}} = \{(x_i) : x_i = 1, \ldots, n\}.$$

For $k \in \mathbb{N}$ and $\sigma = (\sigma_0, \ldots, \sigma_k) \in \{1, \ldots, n\}^{k+1}$, set

$$Z(\sigma) = \{x \in X : x_i = \sigma_i \text{ for } i = 0, \ldots, k\}.$$

We define a metric ρ on X so that $\operatorname{diam} Z(\sigma) = r_\sigma$ where $r_\sigma = \prod_i r_{\sigma_i}$. Thus for $x \neq y$:

$$\rho(x, y) := \begin{cases} \inf\{r_\sigma : x, y \in Z(\sigma)\} & \text{if } x_0 = y_0 \\ 1 & \text{else.} \end{cases}$$

For $j = 1, \dots, n$ define $\theta_j : X \to X$ by

$$\theta_j(x)_i = \begin{cases} j & \text{if } i = 0 \\ x_{i-1} & \text{else.} \end{cases}$$

Note that these maps endow (X, ρ) with metric self-similarity in the sense that

$$\rho(\theta_j(x), \theta_j(y)) = r_j \rho(x, y).$$

Fact [Ed]: The Hausdorff dimension $\dim X$ is the unique number s which satisfies the equation

$$\sum_{j=1}^{n} r_j^s = 1.$$

Moreover, the Hausdorff measure μ^s is determined by

$$\mu^s(Z(\sigma)) = r_\sigma^s.$$

Cuntz algebra.

Let $\mathcal{O}_n$ be the Cuntz algebra generated by n isometries $S_1, \ldots, S_n$ and consider the strongly continuous action $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathcal{O}_n)$ given by

$$\alpha_t(S_j) = r_j^{-it} S_j.$$

Fact (cf. [Ev]): There is a KMS state for α at inverse temperature s iff

$$\sum_{j=1}^{n} r_j^s = 1.$$

Moreover, the KMS state ω is unique and its restriction to $C(X)$ is given by

$$\omega(f) = \int f \, d\mu^s.$$

Question: What is the connection between these two facts?

Groupoids and cocycles

Let Γ be a locally compact Hausdorff étale groupoid and let $c \in Z^1(\Gamma, \mathbb{R})$ be a continuous real-valued cocycle. Then there is a strongly continuous action $\alpha^c : \mathbb{R} \to \operatorname{Aut} C^*(\Gamma)$ such that

$$\alpha_t^c(f)(\gamma) = e^{itc(\gamma)} f(\gamma)$$

for all $f \in C_c(\Gamma)$.

Let μ be a probability measure on Γ^0; then we define a state ω_μ by

$$\omega_\mu(a) = \int P(a)\, d\mu, \quad \text{for } a \in C^*(\Gamma),$$

where $P : C^*(\Gamma) \to C_0(\Gamma^0)$ is the canonical conditional expectation.

If μ is quasi-invariant, the Radon-Nikodym derivative $dr^*\mu/ds^*\mu$ defines a multiplicative 1-cocycle.

KMS States and Cocycles.

Renault proved (see [R, II.5.4]) that if Γ is principal and $c \in Z^1(\Gamma, \mathbb{R})$, then there is a bijective correspondence between KMS states for the associated action α^c and quasi-invariant measures with closely related Radon-Nikodym derivatives.

This was generalized in joint work with Renault (cf. [KR, 3.2]).

Theorem

Let Γ *be an étale groupoid, let* $\beta \in \mathbb{R}$ *and let* $c \in Z^1(\Gamma, \mathbb{R})$ *be such that* $c^{-1}(0)$ *is principal. If* μ *is a quasi-invariant probability measure on* Γ^0 *with Radon-Nikodym derivative*

$$\frac{dr^*\mu}{ds^*\mu} = e^{-\beta c},$$

then ω_μ *is an* (α^c, β)*-KMS state.*

Moreover, every (α^c, β)*-KMS state is of this form.*

Renault-Deaconu Groupoids.

We apply the previous theorem to the groupoid associated to a local homeomorphism (see [R], [D]).

Let $T : X \to X$ be a local homeomorphism. Set:

$$\Gamma := \{(x, m-n, y) : T^m x = T^n y\} \subset X \times \mathbb{Z} \times X.$$

Example: Let $X = \{1, \ldots, n\}^{\mathbb{N}}$ and let T be the shift, $T(x)_i = x_{i+1}$.

Note: Let $\varphi : X \to \mathbb{R}$ be a continuous function and define $c_\varphi : \Gamma \to \mathbb{R}$ by

$$c_\varphi(x, m-n, y) = \sum_{i=0}^{m-1} \varphi(T^i x) - \sum_{j=0}^{n-1} \varphi(T^j y).$$

Then $c_\varphi \in Z^1(\Gamma, \mathbb{R})$ and every $c \in Z^1(\Gamma, \mathbb{R})$ is of this form (see [DKM]).

Main Result.

Theorem

Let (X, ρ) be a locally compact metric space and let $T : X \to X$ be a local homeomorphism. Suppose that there is a continuous real-valued function $\varphi : X \to \mathbb{R}$ such that for all $x \in X$

$$e^{\varphi(x)} = \lim_{y \to x} \frac{\rho(Tx, Ty)}{\rho(x, y)}.$$

Then ω_μ is an (α, β)-KMS state where

$$\alpha = \alpha_{c_\varphi},\ \beta = \dim X \ \textit{and}\ \mu = \mu^\beta / \mu^\beta(X).$$

Proof.

Since $\beta = \dim X$ and μ is (normalized) Hausdorff measure,

$$\frac{dT^*\mu}{d\mu} = e^{-\beta\varphi}.$$

Example and uniqueness.

Example: Let (X, ρ) be as above and let T be the shift. Define $\varphi : X \to \mathbb{R}$ by $\varphi(x) = -\log r_{x_0}$. Then the above result applies to give Evans' KMS state.

Henceforth, suppose X is compact. If φ is positive and T is exact, there is a unique β.

Suppose that T is exact and that φ is both positive and Hölder. By Walters' Theorem there is a unique quasi-invariant probability measure with prescribed Radon-Nikodym derivative (see [W78], [KR, 2.8]).

- T is *exact* if for every nonempty open set $U \subset X$ there is $n \geq 0$ such that $T^n(U) = X$.
- φ is *Hölder* if there are positive constants k and ℓ such that for all $x, y \in X$
$$|\varphi(x) - \varphi(y)| \leq k\rho(x, y)^{\ell}.$$

These conditions hold in the above example.

The uniqueness of the KMS state follows by [KR, 3.2].

Notes.

Note that the positivity of φ ensures that T is positively expansive.

T is said to be *positively expansive* if there is $\varepsilon > 0$ such that

$$\rho(T^n x, T^n y) < \varepsilon \quad \text{for all } n \geq 0$$

implies $x = y$.

Under the above assumptions we have $P(T, -\beta\varphi) = 0$ where $P(T, \cdot)$ is the topological pressure (see [KR, 3.5]).

This notion generalizes topological entropy $h(T)$ in the sense that $h(T) = P(T, 0)$.

If φ is constant then by standard properties of pressure we have

$$h(T) - \beta\varphi = P(T, 0) - \beta\varphi = P(T, -\beta\varphi) = 0;$$

Hence, $h(T) = \beta\varphi$. We will return to this shortly.

Coverings of $\mathbb{T}$.

Let $f : [0,1] \to \mathbb{R}$ be a positive continuous function such that $f(0) = f(1)$ and $n := \int_0^1 f(t)\,dt$ is a positive integer with $n \geq 2$.
Set $X := \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let ρ be the induced metric. Define $T : X \to X$ by

$$T(x) := \int_0^x f(t)\,dt.$$

Then T is an n-fold covering map. Note that

$$f(x) = \lim_{y \longrightarrow x} \frac{\rho(Tx, Ty)}{\rho(x,y)}.$$

So the main result applies with $\varphi = \log f$, $\beta = \dim X = 1$ and $\mu = \mu^1$.
If f is continuously differentiable and $f(x) > 1$ for all $x \in X$ (e.g. $f \equiv n$), then $\varphi = \log f$ is both positive and Hölder, T is exact and, hence, ω_μ is the unique KMS state.

Topological Entropy.

Let $T : X \to X$ be a local homeomorphism. Suppose that (X, ρ) is compact, T is exact and there is a constant $\tau > 1$ such that for all $x \in X$

$$\tau = \lim_{y \longrightarrow x} \frac{\rho(Tx, Ty)}{\rho(x, y)}.$$

Then our results apply to give a unique KMS state ω_μ.

Moreover, as above we have

$$h(T) = \beta \log \tau$$

where $\beta = \dim X$ and $h(T)$ is the topological entropy.

In the above example with $f \equiv n$ we have $\beta = 1$ and $h(T) = \log n$.

Similar results hold for the first example with arbitrary $\beta = s > 0$.

Finite Graphs.

The path space of a finite graph provides another example (see [Ed: §6.4]).

Let $E = (E^1, E^0)$ be a finite graph with primitive vertex matrix A_E and let $r \in (0,1)$. (A_E irreducible and non-cyclic is OK.)

Then there is a metric ρ on E^∞ such that for all $e \in E^1$ and $x, y \in s(e)E^\infty$ we have

$$\rho(ex, ey) = r\rho(x, y).$$

Hence, the shift map T satisfies the local scaling property with $\tau = 1/r$.

Moreover, the Hausdorff dimension s satisfies $\lambda = r^{-s}$ where λ is the Perron-Frobenius eigenvalue of A_E.

If q is the unique eigenvector associated to λ with $\sum_v q_v = 1$, the metric is specified by prescribing the diameter of cylinder sets as follows:

$$\operatorname{diam} Z(\alpha) = q_{s(\alpha)}^{1/s} r^{d(\alpha)}$$

where α is a path of length $d(\alpha)$.

Some references.

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Thanks!

Any questions?