

On k -morphs.

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What are k -morphs?

Many interesting C^* -algebras arise from the study of k -graphs. Morphs provide a way of combining k -graphs.

- A k -morph between two k -graphs yields a $(k + 1)$ -graph.
- Isomorphism classes of k -morphs form a category \mathcal{M}_k .
- Let \mathcal{C}^* be the category of C^* -correspondences. Then there is a functor $\mathcal{M}_k^{\boxtimes} \rightarrow \mathcal{C}^*$.



k -graphs.

Definition: Let Λ be a countable small category and let $d : \Lambda \rightarrow \mathbb{N}^k$ be a functor. We say (Λ, d) is a k -graph, if for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that

$$d(\lambda) = m + n$$

there exist unique $\mu, \nu \in \Lambda$ satisfying:

- $d(\mu) = m$ and $d(\nu) = n$,
- $\lambda = \mu\nu$.

Notation: Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \text{Obj}(\Lambda)$.



Remarks.

Let Λ be a k -graph.

- If $k = 0$, then d is trivial and Λ is just a set.
- If $k = 1$, then Λ is the path category of a directed graph.
- If $k \geq 2$, think of Λ as generated by k graphs of different colors that share the same set of vertices Λ^0 .

Commuting squares, cubes etc. form an essential piece of structure in the last case.



The C^* -algebra $C^*(\Lambda)$.

Suppose $(*) \forall v \in \Lambda^0, n \in \mathbb{N}^k, v\Lambda^n$ is finite and nonempty where

$$v\Lambda^n := r^{-1}(v) \cap \Lambda^n.$$

Definition: Let $C^*(\Lambda)$ be the universal C^* -algebra generated by the set $\{s_\lambda : \lambda \in \Lambda\}$ satisfying:

- for $s(\lambda) = v, s_\lambda^* s_\lambda = s_v,$
- for $v \in \Lambda^0, n \in \mathbb{N}^k,$

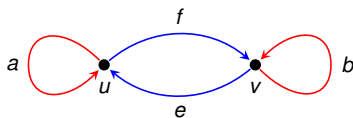
$$s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*,$$

- for $\lambda, \mu \in \Lambda,$

$$s_\lambda s_\mu = \begin{cases} s_{\lambda\mu} & \text{if } s(\lambda) = r(\mu), \\ 0 & \text{else.} \end{cases}$$

Example of a 2-graph.

The following is a representation of a 2-graph Λ . Only the morphisms of minimal degree, $\Lambda^{(1,0)}$ and $\Lambda^{(0,1)}$, are presented. The commutation rules are uniquely determined.



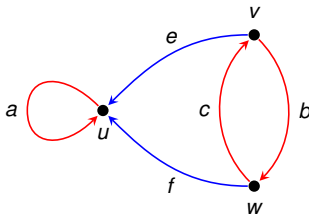
We have

$$C^*(\Lambda) \cong M_2 \otimes C(\mathbb{T}^2).$$



Another Example of a 2-graph.

Here is another 2-graph Σ . Only $\Sigma^{(1,0)}$ and $\Sigma^{(0,1)}$ are given.



We have

$$C^*(\Sigma) \cong M_4 \otimes C(\mathbb{T}).$$



Definition.

Let Λ and Γ be k -graphs. Let X be a countable set and let $r : X \rightarrow \Lambda^0$, $s : X \rightarrow \Gamma^0$ and $\varphi : X * \Gamma \rightarrow \Lambda * X$ be given.

Suppose:

- φ is a bijection,
- if $\varphi(x_1, \gamma_1) = (\lambda_1, x_2)$, the following holds:
 - $d(\gamma_1) = d(\lambda_1)$, $r(x_1) = r(\lambda_1)$ and $s(\gamma_1) = s(x_2)$,
 - If $\varphi(x_2, \gamma_2) = (\lambda_2, x_3)$, then $\varphi(x_1, \gamma_1\gamma_2) = (\lambda_1\lambda_2, x_3)$.

Then the quadruple (X, r, s, φ) is said to be a Λ - Γ *morph*.

Often, we just say that X is a k -*morph*.

If $\Lambda = \Gamma$, say X is a Λ -*endomorph*.



Linking Graph.

For a Λ - Γ morph X , there is a $(k + 1)$ -graph Σ s.t.

- $\Sigma^{(n,0)} = \Lambda^n \sqcup \Gamma^n$ for $n \in \mathbb{N}^k$,
- $\Sigma^{(0,1)} = X$,
- under these identifications,

$$x_1 \gamma = \lambda x_2 \quad \text{iff} \quad \varphi(x_1, \gamma) = (\lambda, x_2).$$

Every k -morph arises in this way. The second example above is a linking graph for the 1-morph $X = \{e, f\}$.



The C^* -correspondence $\mathcal{H}(X)$.

Under mild assumptions on Λ , Γ and X , $C^*(\Sigma)$ makes sense.
Set

$$P := \sum_{u \in \Lambda^0} s_u \quad \text{and} \quad Q := \sum_{v \in \Gamma^0} s_v.$$

Proposition: With notation as above, we may regard

$$\mathcal{H}(X) := PC^*(\Sigma)Q$$

a $C^*(\Lambda)$ - $C^*(\Gamma)$ correspondence; moreover, the left action induces an embedding $C^*(\Lambda) \hookrightarrow \mathcal{K}(\mathcal{H}(X))$.



Sketch of proof.

We assume that both Λ, Γ satisfy $(*)$ and that the structure maps r, s are surjective on X while r is finite-to-one.

For $\lambda, \mu \in \Sigma$ with $s(\lambda) = s(\mu)$ we have

$$s_\lambda s_\mu^* \in QC^*(\Sigma)Q \quad \text{iff} \quad \lambda, \mu \in \Gamma.$$

It follows that $QC^*(\Sigma)Q = C^*(\Gamma)$ and $\mathcal{H}(X)$ is a right Hilbert $C^*(\Gamma)$ -module. Moreover, $\mathcal{K}(\mathcal{H}(X)) \cong PC^*(\Sigma)P$.

The embedding $C^*(\Lambda) \hookrightarrow PC^*(\Sigma)P$ combined with the above isomorphism yields the desired embedding

$$C^*(\Lambda) \hookrightarrow \mathcal{K}(\mathcal{H}(X)).$$



Composing Morphs.

Given a Λ - Γ morph X and a Γ - Σ morph Y ,
we construct a Λ - Γ morph

$$Z = X * Y = \{(x, y) : s(x) = r(y)\},$$

by defining

$$\varphi_Z(x, y, \sigma) = (\lambda, x', y'),$$

if there is $\gamma \in \Gamma$ such that

$$\varphi_X(x, \gamma) = (\lambda, x'),$$

$$\varphi_Y(y, \sigma) = (\gamma, y').$$



The Category \mathcal{M}_k .

This composition allows one to define the category \mathcal{M}_k as follows:

- $\text{Obj}(\mathcal{M}_k)$ consists of k graphs,
- $\text{Mor}_{\mathcal{M}_k}(\Gamma, \Lambda) = \text{iso. classes of } \Lambda\text{-}\Gamma \text{ morphs,}$
- composition is given by

$$[X] \circ [Y] = [X * Y].$$



The Subcategory $\mathcal{M}_k^{\boxtimes}$.

The subcategory $\mathcal{M}_k^{\boxtimes}$ consists of k -graphs satisfying $(*)$ and k -morphs satisfying a mild condition.

Lemma: Given a Λ - Γ morph X and a Γ - Σ morph Y in $\mathcal{M}_k^{\boxtimes}$:

$$\mathcal{H}(X * Y) \cong \mathcal{H}(X) \otimes_{C^*(\Gamma)} \mathcal{H}(Y).$$

Let \mathcal{C}^* be the category of C^* -algebras and iso. classes of C^* -correspondences.

Theorem: There is a functor from $\mathcal{M}_k^{\boxtimes}$ to \mathcal{C}^* given by

$$\begin{aligned} \Lambda &\mapsto C^*(\Lambda), \\ [X] &\mapsto [\mathcal{H}(X)]. \end{aligned}$$



Endomorphs.

For a Λ -endomorph X , there is a $(k + 1)$ -graph $\Sigma = \Lambda \times_X \mathbb{N}$ s.t.

- $\Sigma^{(n,0)} = \Lambda^n$ and $\Sigma^{(0,1)} = X$,
- under these identifications,

$$x_1 \lambda_1 = \lambda_2 x_2 \quad \text{iff} \quad \varphi(x_1, \lambda_1) = (\lambda_2, x_2).$$

Then $\Lambda \times_X \mathbb{N}$ is called the *endomorph skew-graph*.

Theorem: For X and Λ in $\mathcal{M}_k^{\mathbb{X}}$ we have:

$$C^*(\Lambda \times_X \mathbb{N}) \cong \mathcal{O}_{\mathcal{H}(X)}.$$



Thanks!
Any questions?