Chapter 5: Confidence Intervals

Section 5.1: Large sample confidence interval for the mean
Introduction

- **Problem:** point estimates are almost never exactly equal to the true values of the parameters they are estimating (sampling variability).

- **Practical Point:** need to know just how far off from the true value the point estimate is likely to be.

- How to describe **accuracy** of a point estimate: use its standard deviation, or uncertainty.
Example

Assume that a large number of independent measurements from a normal population, are made on the diameter of a piston. The sample mean of the measurements is 14.0 cm, and the uncertainty in this quantity, which is the standard deviation of the sample mean, is 0.1 cm.

Take an interval: $\bar{X} \pm 3\sigma_{\bar{X}} = 14 \pm 3 \times 0.1 = (13.7, 14.3)$.

So, we have a high level of confidence that the true diameter is in the interval $(13.7, 14.3)$. This is because it is highly unlikely that the sample mean will differ from the true diameter by more than three standard deviations.
Think of the piston diameter example:

Since the population mean will not be exactly equal to the sample mean of 14, it is best to construct a confidence interval around 14 that is likely to cover the population mean.

We can then quantify our level of confidence that the population mean is actually covered by the interval.
Constructing a CI

Let \( \mu \) be the unknown population mean and let \( \sigma^2 \) be the unknown population variance.

Let \( X_1, \ldots, X_{100} \) (large sample) be the 100 diameters of the pistons.

Compute the sample mean: \( \bar{X} \)

Then \( \bar{X} \sim N(\mu, \sigma / \sqrt{100}) \) by the CLT.
CI does or does not cover the true mean $\mu$

Consider the distribution of $\bar{X}$

The middle 95% of the $\bar{X}$ pdf curve extends $1.96\sigma_{\bar{X}}$ on either side of the population mean $\mu$.

Look at the interval: $\bar{X} \pm 1.96\sigma_{\bar{X}}$

It covers or does not cover $\mu$ depending on the value of $\bar{X}$
Computing a 95% CI for the population mean $\mu$

The 95% confidence interval (CI) is $\bar{X} \pm 1.96\sigma_{\bar{X}}$.

Example: Piston problem: A 95% CI for the mean is $14 \pm 1.96(0.1)$.

We can use the sample standard deviation as an estimate for the population standard deviation, since the sample size is large.

We can say that we are 95% confident, or confident at the 95% level, that the population mean diameter for pistons lies, between 13.804 and 14.196.

Warning: The methods described here require that the data be a random sample from a population. When used for other samples, the results may not be meaningful.
Does this 95% confidence interval actually cover the population mean $\mu$?

Not always! See the graphs on the previous slide – it depends on the sample (via the value of $\bar{X}$).

In practice we do not know if the CI we constructed based on our sample does or does not cover the mean $\mu$.

In the long run, if we repeatedly constructed these confidence intervals, then 95% of the CIs will cover the true mean $\mu$. 
Other Confidence Levels: the values $z_{\alpha/2}$

Let confidence level $C = 1 - \alpha$, the corresponding $z$-value is commonly denoted by $z_{\alpha/2}$.

$z_{\alpha/2}$ cuts probability $\alpha/2$ to its right under the standard normal curve.
EXAMPLES OF $z_{\alpha/2}$.

- Take confidence level $C=90\%$. Find $z_{\alpha/2}$.
  
  $C=0.90$, so $\alpha=0.1$  $\alpha/2=0.05$. From Z-table $z_{0.025}=1.645$.

- Take confidence level $99\%$. Find $z_{\alpha/2}$.
  
  $C=0.99$, so $\alpha=0.01$  $\alpha/2=0.005$. From Z-table $z_{0.005}=2.575$. 
General 100(1 − \(\alpha\))% CI

Let \(X_1, \ldots, X_n\) be a large \((n > 30)\) random sample from a population with mean \(\mu\) and standard deviation \(\sigma\), so that \(\bar{X}\) is approximately normal.

Then a level 100(1 − \(\alpha\))% confidence interval for \(\mu\) is

\[
\bar{X} \pm z_{\alpha/2} \sigma / \sqrt{n}
\]

When the value of \(\sigma\) is unknown, it can be replaced with the sample standard deviation \(s\).
Specific Confidence Intervals for $\mu$

- $\bar{X} \pm \frac{s}{\sqrt{n}}$ is a 68% interval for $\mu$.
- $\bar{X} \pm 1.645 \frac{s}{\sqrt{n}}$ is a 90% interval for $\mu$.
- $\bar{X} \pm 1.96 \frac{s}{\sqrt{n}}$ is a 95% interval for $\mu$.
- $\bar{X} \pm 2.58 \frac{s}{\sqrt{n}}$ is a 99% interval for $\mu$.
- $\bar{X} \pm 3 \frac{s}{\sqrt{n}}$ is a 99.7% interval for $\mu$. 
Example

The sample mean for the fill weights of 100 boxes is 12.05, and the standard deviation is \( s = 0.1 \). Find an 85% confidence interval for the mean fill weight of the boxes.
Example

There is a sample of 50 microdrills with an average lifetime (expressed as the number of holes drilled before failure) of 12.68 and a standard deviation of 6.83. Suppose an engineer reported a confidence interval of (11.09, 14.27) but neglected to specify the level. What is the level of this confidence interval?
Probability vs. Confidence

- In computing a CI, such as the one of diameter of pistons: (13.804, 14.196), it is tempting to say that the probability that $\mu$ lies in this interval is 95%. Do not be tempted!

- The term probability refers to random events, which can come out differently when experiments are repeated.

- The numbers 13.804 and 14.196 are fixed, not random. The population mean is also fixed. The mean diameter is either in the interval or not. There is no randomness involved.

- So, we say that we have 95% confidence (not probability) that the population mean is in this interval.
Example

A 90% confidence interval for the mean diameter (in cm) of steel rods manufactured on a certain extrusion machine is computed to be (14.73, 14.91).

True or false: The probability that the mean diameter of rods manufactured by this process is between 14.73 and 14.91 is 90%.
Determining Sample Size for a Confidence Interval of Specified Width

Back to the example of diameter of pistons: We had a CI of (13.804, 14.196).

This interval specifies the mean to within ±0.196. Now assume that the interval is too wide to be useful.

Assume that it is desirable to produce a 95% confidence interval that specifies the mean to within ± 0.1.

How to do that?
Choice of the sample size for a given margin of error.

Let \( X_1, X_2, \ldots, X_n \) sample from \( N(\mu, \sigma) \), \( \mu \) unknown, \( \sigma \) known.

The margin of error \( m = \) half-length of the CI for \( \mu \), \( m = \) depends on:

- the **confidence level** via \( Z_{\alpha/2} \) (as conf. level \( \uparrow \), \( m \uparrow \));
- the **variability** in the population \( \sigma \) (as \( \sigma \uparrow \), \( m \uparrow \));
- the **sample size** (as \( n \uparrow \), \( m \downarrow \)).

To decrease the error, but keep the confidence level unchanged, we need to increase the sample size.

For a \((1-\alpha)\) CI for \( \mu \) (\( \sigma \) known) to have margin of error \( m \), we need sample size

\[
   n = \frac{Z_{\alpha/2}^2 \sigma^2}{m^2}.
\]
Example

Back to the pistons.
We had a CI of (13.804, 14.196), so margin of error \( m = \pm 0.196 \).

Now we want a 95% CI that specifies mean to within \( \pm 0.1 \). That is the margin of error is 0.1.

To do this, the sample size must be increased to

\[
\frac{Z_{\alpha/2}^2 \sigma^2}{m^2}.
\]

Substituting for \( z \), \( \sigma \) and \( m \) we get:

\[
n \geq \frac{1.96^2 \cdot 1^2}{0.1^2} = 384.6, \text{ ROUND UP to } n=385.
\]

The smallest \( n \) that guarantees the margin of error at most 0.1 is \( n=385 \).

**NOTE:** When looking for \( n \), always round up!
Example

Randomly selected students from a university participated in an experiment to test their ability to determine when 1 min (60 seconds) has passed. 40 students yielded a sample mean of 58.3 sec. Assuming that $\sigma = 9.5$ sec, construct a 95% CI for the population mean time for all students in that university.

Solution: $z_{\alpha/2} = 1.96$, $\bar{X} = 58.3$ sec, $\sigma = 9.5$ sec.

95% CI for the true mean perception time of 1 min:

$$(\bar{X} \pm 1.96\sigma / \sqrt{n}) = 58.3 \pm 1.96 \times 9.5 / \sqrt{40} = (55.356, 61.244).$$

The margin of error for this estimate is: 2.9

How many students should be sampled, so that the margin of error is less than 1 sec., with the same confidence level?

$$n = \frac{z_{\alpha/2}^2 \sigma^2}{m^2} = \frac{1.96^2 \times 9.5^2}{1^2} = 346.7 \approx 367$$

What if we lower the confidence level to 80%? $z_{\alpha/2} = z_{0.1} = 1.28$

$$n = \frac{z_{\alpha/2}^2 \sigma^2}{m^2} = \frac{1.28^2 \times 9.5^2}{1^2} = 147.9 \approx 148$$

NOTE: Lowering the confidence level allows for a smaller sample size.