Sections 5.2 and 5.3.

Large sample CI for a proportion and small sample CI for a mean.
5.2: Confidence Interval for a Proportion

Estimating proportion of successes in a binomial experiment

**Binomial experiment**, $X =$ number of successes, $n =$ number of trials, $p =$ probability of success is unknown.

Take $\hat{p} = \frac{X}{n} = \frac{\text{number of successes}}{\text{number of trials}} =$ sample proportion of successes.

Then, $\hat{p}$ is an **unbiased point** estimator of $p$.

How to get an interval estimate of $p$?

We start with a large sample and use the Central Limit Theorem.
95% large sample CI for $p$

When $n$ is large, the probability is 0.95 that the sample proportion is within 1.96 standard deviations of the true proportion (using normal approximation):

$$p - 1.96 \sqrt{\frac{p(1-p)}{n}} < \hat{p} < p + 1.96 \sqrt{\frac{p(1-p)}{n}}$$

It is then also true that for 95% of all possible samples above inequality works.

So, the above interval is a great candidate for the 95% CI for $p$. 
Problem: Consider the formula for the interval we got:

\[ \hat{p} - 1.96 \sqrt{\frac{p(1-p)}{n}} < p < \hat{p} + 1.96 \sqrt{\frac{p(1-p)}{n}} \]

This expression is not a practical confidence interval, because it contains the unknown population proportion \( p \) in the margin of error.

The traditional approach is to replace \( p \) with \( \hat{p} \).

Recent research shows that a slight modification of \( n \) and the following estimate of \( p \) provide a good confidence interval:

Define \( \tilde{n} = n + 4 \) and \( \tilde{p} = \frac{X + 2}{\tilde{n}} \).
Confidence Interval for $p$

Let $X$ be the number of successes in $n$ independent Bernoulli trials with success probability $p$, so that $X \sim \text{Bin}(n, p)$.

Then a $100(1 - \alpha)\%$ confidence interval for $p$ is

$$
\tilde{p} \pm z_{\alpha/2} \sqrt{\frac{\tilde{p}(1 - \tilde{p})}{\tilde{n}}}
$$

Where $\tilde{n} = n + 4$ and $\tilde{p} = \frac{X + 2}{\tilde{n}}$

If the lower limit is less than 0, replace it with 0.

If the upper limit is greater than 1, replace it with 1.
Example

It was reported that, in a sample of 507 adult Americans, only 142 correctly described the Bill of Rights as the first ten amendments to the U.S. Constitution. Calculate a 99% CI for the proportion of all U. S. adults that could give a correct description of the Bill of Rights.
Sample size for given margin of error

Suppose we want to estimate proportion to within margin of error \( m \). How large a sample do we need?

\[
n = \frac{z_{\alpha/2}^2 \hat{p}(1 - \hat{p})}{m^2} - 4
\]

Example: What sample size is needed to obtain a 99\% confidence interval for the of all U. S. adults that could give a correct description of the Bill of Rights with width (margin of error) \( \pm 0.01 \)?

Example: What if we did not have any prior info on \( p \)? What sample size do we need then?

NOTE: \( \hat{p}(1 - \hat{p}) \) is maximized for \( \hat{p} = 0.5 \). To get (conservative) sample size, use \( \hat{p} = 0.5 \) is the above formula for \( n \) to get:

\[
n = \frac{z_{\alpha/2}^2}{4m^2} - 4
\]
The Traditional Method for CI for p

- Let \( \hat{p} \) be the proportion of successes in a large number of independent Bernoulli trials with success probability \( p \).

- Then the traditional level 100\((1 - \alpha)\)% confidence interval for \( p \) is

\[
\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}.
\]

- The method should not be used unless the sample contains at least 19 successes and 10 failures.

- To obtain a \((1 - \alpha)100\)% CI for proportion using this method, that has margin of error =\( m \), we need sample size

\[
n = \frac{z_{\alpha/2}^2 \hat{p}(1 - \hat{p})}{m^2}
\]

- If we do not have any prior estimate of \( p \), use \( \hat{p} = 0.5 \) in the formula for the sample size, you get:

\[
n = \frac{z_{\alpha/2}^2}{4m^2}
\]
Traditional method of estimating \( p \).

**Example:** use the data on the Bill of Rights study.

Use the traditional method to find 99% CI for \( p \), and find the sample size needed to obtain a 99% confidence interval for \( p \) with width (margin of error) \( \pm 0.01 \)?

Also, assuming that we do not have any estimates of \( p \) available, estimate the sample size needed to obtain a 99% confidence interval for \( p \) with width (margin of error) \( \pm 0.01 \)?

Compare the results to those obtained using the previous method.
5.3: Small Sample CIs for a Population Mean

- The methods that we have discussed for a population mean previously require that the sample size be large.

- When the sample size is small, there are no good general methods for finding CIs.

- However, when the population is approximately normal, a probability distribution called the Student’s $t$ distribution can be used to compute confidence intervals for a population mean.
Small-Sample Confidence Intervals for the Mean

- What can we do if $\bar{X}$ is the mean of a *small* sample?

- If the sample size is small, $s$ may not be close to $\sigma$, and $\bar{X}$ may not be approximately normal. If we know nothing about the population from which the small sample was drawn, there are no easy methods for computing CIs.

- If the population is approximately normal, it will be approximately normal even when the sample size is small. It turns out that we can use the quantity

$$\frac{(\bar{X} - \mu)}{(s / \sqrt{n})}$$

but since $s$ may not be close to $\sigma$, this quantity has a Student’s $t$ distribution.
Student’s $t$ Distribution

- Let $X_1, \ldots, X_n$ be a small ($n < 30$) random sample from a normal population with mean $\mu$. Then the quantity
  \[
  \frac{\bar{X} - \mu}{s / \sqrt{n}}
  \]
  has a Student’s $t$ distribution with $n - 1$ degrees of freedom (denoted by $t_{n-1}$).

- When $n$ is large, the distribution of the above quantity is very close to normal, so the normal curve can be used, rather than the Student’s $t$. 

More on Student’s $t$

- The probability density of the Student’s $t$ distribution is different for different degrees of freedom.

- The $t$ curves are more spread out than the standard normal distribution.

Table A.3, called a $t$ table, provides probabilities associated with the Student’s $t$ distribution.
Example: using t-table

Find the value for the $t_{14}$ distribution whose lower-tail probability is 0.01.

Soln: Look down the column headed with “0.01” to the row corresponding to 14 degrees of freedom. The value for $t = 2.624$. This value cuts off an area, or probability, of 1% in the upper tail. The value whose lower-tail probability is 1% is $-2.624$. 
Student’s $t$ CI for the mean: normal population, $\sigma$ not known

Let $X_1, \ldots, X_n$ be a small random sample from a normal population with mean $\mu$. Then a level 100$(1 - \alpha)$% CI for $\mu$ is

$$
\bar{X} \pm t_{n-1,\alpha/2} \frac{S}{\sqrt{n}}.
$$

The sample must come from a population that is approximately normal.

Note: Normal or approximately normal samples are roughly symmetric and (practically) do not contain outliers.

Other CIs: normal population, $\sigma$ known

If a small sample is taken from a normal population with standard deviation $\sigma$ known, then we use the CI that is using the $z$ value.
Example 8

A random sample of $n = 8$ E-glass fiber test specimens of a certain type yielded a sample mean interfacial shear yield stress of 30.5 and a sample standard deviation of 3.0. Assuming that interfacial shear yield stress is normally distributed, compute a 95% CI for true average stress?
Example

The article “Direct Strut-and-Tie Model for Prestressed Deep Beams” presents measurements of the nominal shear strength (in kN) for a sample of 15 prestressed concrete beams. The results are

580   400   428   825   850   875   920   550 575   750   636   360   590   735   950

Assume that on the basis of a very large number of previous measurements of other beams, the population of shear strengths is known to be approximately normal, with standard deviation $\sigma = 180.0$ kN. Find a 99% confidence interval for the mean shear strength.

MINITAB exercise.

One-Sample Z: strength

The assumed standard deviation = 180

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>SE Mean</th>
<th>99% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>strength</td>
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<td>668.3</td>
<td>192.1</td>
<td>46.5</td>
<td>(548.6, 788.0)</td>
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</tbody>
</table>

What is SE Mean in MINITAB output? It is $\sigma \bar{X} = \text{StDev}/\sqrt{N}$

What if we did not have the population standard deviation $\sigma$?

One-Sample T: strength

<table>
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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>strength</td>
<td>15</td>
<td>668.3</td>
<td>192.1</td>
<td>49.6</td>
<td>(520.6, 815.9)</td>
</tr>
</tbody>
</table>
MINITAB: computing CI for p.

It was reported that, in a sample of 507 adult Americans, only 142 correctly described the Bill of Rights as the first ten amendments to the U.S. Constitution. Calculate a 99% CI for the proportion of all U. S. adults that could give a correct description of the Bill of Rights.

Test and CI for One Proportion

<table>
<thead>
<tr>
<th>Sample</th>
<th>X</th>
<th>N</th>
<th>Sample p</th>
<th>99% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>142</td>
<td>507</td>
<td>0.280079</td>
<td>(0.230011, 0.334362)</td>
</tr>
</tbody>
</table>