Lecture 3. PROBABILITY

Sections 2.1 and 2.2

Experiment, sample space, events, probability axioms.

Counting techniques
The probability theory begins in attempts to describe gambling (how to win, how to divide the stakes, etc.), probability theory mainly considered discrete events, and its methods were mainly combinatorial

Gerolamo Cardano (September 24, 1501 – September 21, 1576)

Author of the first book on probability “De Ludo Aleae” ~ “On the dice game” written in 1560s, published in 1663
... eventually, analytical considerations motivated the incorporation of continuous variables into the theory. The foundations of modern theory of probability were laid by Andrey Nikolaevich Kolmogorov, who combined the notion of sample space, introduced by Richard von Mises, and Lebesgue measure theory and presented his axiom system for probability theory in 1933 (Grundbegriffe der Wahrscheinlichkeitsrechnung, by A. Kolmogorov, Julius Springer, Berlin, 1933, 62 pp.)

Henri Léon Lebesgue
(June 28, 1875 – July 26, 1941)

Andrei Kolmogorov (1903-1987):
A founder of modern theory of probabilities (1933)

Richard Edler von Mises
(19 April 1883 - 14 July 1953)
Roots of Probability lie in the Games of chance

**Deck of playing cards**

**Dice**

**Roulette**
PROBABILITY IDEAS

Random Experiment – know all possible outcomes, BUT
  - can not predict the outcome of a particular trial.

SAMPLE SPACE S – set of all possible outcomes of a random experiment

EXAMPLES: Toss a coin: Sample space = \{T, H\}

2. Roll a die, observe the score on top. Sample space = \{1, 2, 3, 4, 5, 6\}.

3. Throw a basketball, record the number of attempts to the first basket. Sample space = \{1, 2, 3, 4, \ldots\}.

4. Wait for a Taxi, record waiting time in seconds. Sample space = \{t: t \geq 0\}, t=time.
EVENTS

Event – a combination of one or more outcomes
- a subset of the sample space.

Events are usually denoted by capital letters: A, B, C, ...

EXAMPLES: Toss a coin: Event A: Head comes up. A= \{ H\}.

2. Roll a die. Event A: Even number comes up. A = \{2, 4, 6\}.
   Event B: Number smaller than 3 comes up. B= \{1, 2\}.

3. Throw a basketball. Event A: Get the basket before 4th trial. A= \{1, 2, 3\}.


IN PROBABILITY WE LOOK FOR PROBABILITIES (CHANCES) OF EVENTS
An interpretation of probability

Random experiment → outcome/event

Q: What are the chances/probability of that event?

• Repeat the experiment many, many times.

• Record the PROPORTION of times that event occurred in the repeated experiments.

• Proportion=relative frequency.

• Probability ≈ long-term relative frequency of an event/outcome.

EXAMPLE. P(H) ≈ proportion of H in many (say 100,000) tosses of a coin.
The AXIOMS OF PROBABILITY

• **Axiom 1.** Probability of an event $A$, $P(A)$, is a number between 0 and 1 (inclusive).

• **Axiom 2.** In any random experiment, probability of the sample space is 1.

• **Axiom 3.** Probability of $(A \text{ or } B)$ is the sum of $P(A)$ and $P(B)$ if events $A$ and $B$ are disjoint, i.e. $P(A \text{ or } B) = P(A) + P(B)$ if $A$ and $B$ do not overlap

More generally, if $A_1$, $A_2$, $A_3$, ... are mutually exclusive events, then

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } ... ) = \sum_{i=1}^{\infty} P(A_i)$$

Venn diagram for two events $A$ and $B$ with no common outcomes, disjoint $A$ and $B$. 

Sample Space

- $A$
- $B$
**PROBABILITY RULES**

- **Rule 1.** Probability that event $A$ does not occur $P(\text{not } A) = P(A^c) = 1 - P(A)$.
- **Rule 2.** Probability of the empty set is zero: $P(\emptyset) = 0$.
- **Rule 3. Addition rule.** For any events $A$ and $B$:
  
  $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Illustrations with Venn diagrams. $S =$Sample space
EXAMPLE

Roll a die twice, record the score on both rolls.  
**Sample space:** 36 ordered pairs

**Event A:** All results with sum of scores equal to 7.

A={(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}

**Event B:** All results with first score equal to 5.

B={(5, 1), (5, 2), ...(5, 6)}.

A or B={(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), (5, 1), (5,3), (5,4), (5,5), (5, 6)}.

A and B={(5,2)}. 

S= {(1, 1), (1, 2), (1, 3), ... , (1,6), (2, 1), (2,2), ..., (2, 6), ........ (6,1), (6, 2), (6, 3), ... , (6, 6)}. 
Example

Let
A be the event that it will rain tomorrow, and
B the event that it will rain on Wednesday, September 5.

• A or B = event that it will rain tomorrow or Wed, Sep 5 or on both days.

• A and B = event that it will rain on both days.
EQUALLY LIKELY OUTCOMES

Fair die ➜ all outcomes/scores have the same chances of occurring.

Fair coin ➜ H and T have 50-50 chances of coming up.

Loaded die ➜ some scores have larger chances of coming up than others.

When all outcomes have the same probability, we say they are equally likely.

In an experiment with finite number of equally likely outcomes, the probability of an outcome is $1/\text{(total number of possible outcomes)}$.

And for an event $A$

$$P(A) = \frac{\text{number of outcomes that make up A}}{\text{total number of possible outcomes}}.$$
Example

Adults are randomly selected for a Gallup poll and asked if they think that cloning of humans should be allowed. Among those surveyed,

• 91 said that cloning should be allowed;
• 901 said is should not be allowed, and
• 20 had no opinion.

Based on these results estimate the probability that a randomly selected person believes that cloning of humans should be allowed.

• Solution. Need to know the sample space, total number of responses:

\[91 + 901 + 20 = 1012.\]

Use relative frequency approach to probability:

Events: A- selected person thinks cloning of humans should be allowed.

\[P(A) = \frac{\text{#people thinking cloning should be allowed}}{\text{total number of people surveyed}} = \frac{91}{1012} = 0.0899.\]

Answer: The probability that a person thinks cloning of humans should be allowed is about 0.09.
Example

• The probability that John will pass a math class is 0.7, and that he will pass a biology class is 0.6. The probability that he will pass both classes is 0.5. What is the probability that he will pass at least one of these classes? What is the probability that he will fail Biology?

• Solution. Events:

M= John will pass math; B=John will pass Biology

P(M)=0.7, P(B)=0.6; P(M and B)=0.5; Need: P(M or B)=?

Addition rule: P(M or B) = P(M) + P(B) + P(M and B)

= 0.7 + 0.6 - 0.5 = 0.8

Probability that John will fail Biology = P( not B) = 1 - P(B) = 1 - 0.6 = 0.4.

• Answer: The probability that John will pass Math or Biology is 0.8. The probability that John will fail Biology is 0.4.
Example:

The following table summarizes data on pedestrian deaths caused by accidents (Natl. Highway Traffic Safety Admin). If one of the pedestrian deaths is randomly selected, what is the probability that:

a. Pedestrian was intoxicated, but the driver was not intoxicated.
b. Pedestrian or driver (at least one) was intoxicated?

<table>
<thead>
<tr>
<th>Driver Intoxicated?</th>
<th>Pedestrian Intoxicated?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>59</td>
<td>79</td>
</tr>
<tr>
<td>266</td>
<td>581</td>
</tr>
</tbody>
</table>

Solution. Total number of people in the study = 59 + 79 + 266 + 581 = 985

a. \( \frac{266}{985} = 0.27 \)

b. W-event that ped. was intox. \( P(W) = \frac{(59 + 266)}{985} = 0.33 \);
   D-event that driver was intox. \( P(D) = \frac{(59 + 79)}{985} = 0.14 \)
   \[ P(W \text{ and } D) = \frac{59}{985} = 0.06 \]
   \[ P(W \text{ or } D) = \text{ via addition rule} = P(W) + P(D) + P(W \text{ and } D) = 0.33 + 0.14 - 0.06 = 0.41 \]
PRINCIPLES OF COUNTING

A certain car is available in 3 colors and 2 engine sizes. In how many ways can a buyer choose a car?

(3 colors)(2 engines) = 6 ways to choose a car

The Fundamental Principle of Counting: If one operation can be performed in $n$ ways and another operation in $m$ ways, then the total number of ways to perform both operations is $nm$.

This can be generalized to $k$ operations (see page 63 in your book).

Example: A computer can be ordered with 3 choices of hard drive, 4 choices of amount of memory, and 2 choices of video card, and 3 choices of the monitor. In how many ways can this computer be ordered?

$3 \times 4 \times 2 \times 3 = 72$ ways
Permutations and Combinations

Suppose you have a set of $n$ different objects.

- **Permutation** is an ordering of the objects in the set. There are $n!$ permutations of the set of $n$ elements.

- **Example:** In how many ways can you order 5 books on a shelf? Answer: $5!$

- A combination of $k$ objects out of a set of $n$ objects is choosing a subset of $k$ objects from the set of $n$ objects. The number of subsets of size $k$ from a set of size $n$ is \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

- **Example:** In how many ways can you choose 7 books out of a box containing 10 books and order them on a shelf? Answer: $\binom{10}{7} \times 7!$