Section 2.5

Linear Functions of Random Variables
Linear Functions of Random Variables

If $X$ is a random variable, and $a$ and $b$ are constants, then

$Y = aX + b$ is a linear function of $X$. $Y$ is also a random variable, and

\[
\mu_{aX+b} = a\mu_X + b
\]

\[
\sigma_{aX+b}^2 = a^2\sigma_X^2
\]

\[
\sigma_{aX+b} = |a|\sigma_X
\]
More Linear Functions

If $X$ and $Y$ are random variables, and $a$ and $b$ are constants, then

$$\mu_{aX+bY} = \mu_{aX} + \mu_{bY} = a\mu_X + b\mu_Y.$$ 

More generally, if $X_1, \ldots, X_n$ are random variables and $c_1, \ldots, c_n$ are constants, then the mean of the linear combination $c_1X_1+\ldots+c_nX_n$ is given by

$$\mu_{c_1X_1+c_2X_2+\ldots+c_nX_n} = c_1\mu_{X_1} + c_2\mu_{X_2} + \ldots + c_n\mu_{X_n}.$$
Two Independent Random Variables

If $X$ and $Y$ are independent random variables, and $S$ and $T$ are sets of numbers, then

$$P(X \in S \text{ and } Y \in T) = P(X \in S)P(Y \in T).$$

More generally, if $X_1, \ldots, X_n$ are independent random variables, and $S_1, \ldots, S_n$ are sets, then

$$P(X_1 \in S_1, X_2 \in S_2, \ldots, X_n \in S_n) = P(X_1 \in S_1)P(X_2 \in S_2) \cdots P(X_n \in S_n).$$
Variance Properties

If $X_1, \ldots, X_n$ are independent random variables, then the variance of the sum $X_1 + \cdots + X_n$ is given by

$$\sigma_{X_1+X_2+\cdots+X_n}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_n}^2.$$  

If $X_1, \ldots, X_n$ are independent random variables and $c_1, \ldots, c_n$ are constants, then the variance of the linear combination $c_1 X_1 + \cdots + c_n X_n$ is given by

$$\sigma_{c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}^2 = c_1^2 \sigma_{X_1}^2 + c_2^2 \sigma_{X_2}^2 + \cdots + c_n^2 \sigma_{X_n}^2.$$  

If $X$ and $Y$ are independent random variables with variances $\sigma_X^2$ and $\sigma_Y^2$, then the variance of the sum $X + Y$ is $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$.  

The variance of the difference $X - Y$ is $\sigma_{X-Y}^2 = \sigma_X^2 + \sigma_Y^2$. 
Example

A piston is placed inside a cylinder. The clearance is the distance between the edge of the piston and the wall of the cylinder and is equal to one-half the difference between the cylinder diameter and the piston diameter. Assume the piston diameter has a mean of 80.85 cm with a standard deviation of 0.02 cm. Assume the cylinder diameter has a mean of 80.95 cm with a standard deviation of 0.03 cm. Find the mean clearance. Assuming that the piston and cylinder are chosen independently, find the standard deviation of the clearance.

Sln: Let $X=$piston diameter, $EX=80.85$ cm, St. Dev$X=0.02$ cm.

$Y=$cylinder diameter, $EY=80.95$, St. dev. $Y= 0.03$cm. $X$ and $Y$ are independent. $C=$clearance$= 0.5(Y – X)$.

$EC= 0.5(EY – EX) = 0.5(80.95 – 80.85)=0.05$cm = average clearance.

$\text{Var}(C)=\text{Var}(0.5(Y – X))= (0.5)^2(\text{Var}X + \text{Var}Y)= (0.25)(0.02^2 + 0.03^2)= 0.000325.$

$\text{St. Dev}(C)=\sqrt{0.000325} = 0.018$ cm.
Independence and Simple Random Samples

Definition: If $X_1, \ldots, X_n$ is a simple random sample, then $X_1, \ldots, X_n$ may be treated as independent random variables, all with the same distribution.

When $X_1, \ldots, X_n$ are independent random variables, all with the same distribution, we sometimes say that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d).
Properties of $\overline{X}$

If $X_1, \ldots, X_n$ is a simple random sample from a population with mean $\mu$ and variance $\sigma^2$, then the sample mean $\overline{X}$ is a random variable with $\mu_{\overline{X}} = \mu$ and variance $\sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$. The standard deviation of $\overline{X}$ is $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$. 
Example

A process that fills plastic bottles with a beverage has a mean fill volume of 2.013 L and a standard deviation of 0.005 L. A case contains 24 bottles. Assuming that the bottles in a case are a simple random sample of bottles filled by this method, find the mean and standard deviation of the average volume per bottle in a case.

**Soln:** $X_1, \ldots, X_{24}$ volume of beverage in the 24 bottles. $X_i$’s have the same distribution and are independent.

$E X_i = 2.013$ L, and $\text{st.dev } X_i = 0.005$ L.

Average volume per bottle $= \bar{X}$, $E \bar{X} = E X_i = 2.013$ L,

$\text{st.dev } \bar{X} = (\text{st.dev } X_i)/\sqrt{n} = 0.005/\sqrt{24} = 0.001$ L