2) \( y' = \frac{\sqrt{y}}{x} \)

\[ xy' - \sqrt{y} = 0 \quad \text{Nonlinear \% of } \sqrt{y} \]

\[ x \frac{dy}{dx} = \sqrt{y} \quad \Rightarrow \quad \frac{dy}{\sqrt{y}} = \frac{dx}{x} = \Rightarrow \text{ separable.} \]

Not autonomous since \( y' \neq f(y) \)

3) \[
\int \frac{dy}{\sqrt{y-1}} = \int \frac{dx}{x^2} \quad y \neq 1
\]

Let \( y-1 = u \)

\( dy = du \)

\[
\int u^{-\frac{1}{2}} du = \int x^{-2} dx
\]

\[
\frac{1}{2} u^{\frac{1}{2}} = \frac{x^{-1}}{-1} + C
\]

\[
2 \sqrt{y-1} = - \frac{1}{x} + C
\]

\[
\sqrt{y-1} = - \frac{1}{2x} + C
\]
\[
\begin{align*}
y - 1 &= \left(C - \frac{1}{2x}\right)^2 \\
y &= 1 + \left(C - \frac{1}{2x}\right)^2
\end{align*}
\]

Check if \( y(x) = 1 \) is a solution to \( \frac{dy}{dx} = \frac{\sqrt{y-1}}{x^2} \):

\[
\begin{align*}
\text{If } y &= 1, \\
y' &= 0 \\
o &= \frac{\sqrt{1-1}}{x^2} \\
\Rightarrow 0 &= 0
\end{align*}
\]

\[\therefore y(x) = 1 \text{ is a solution to ODE.}\]

Since \( y(x) = 1 \) cannot be obtained from the general solution \( y = 1 + \left(C - \frac{1}{2x}\right)^2 \) for any value of \( C \), it follows that

\[y(x) = 1 \text{ is a singular solution.}\]
4) \[ x^2 \frac{dy}{dx} + xy = 1 \quad y(1) = 2 \]  

It is a first order linear ODE.

Write it in standard form: \[ y' + P(x)y = f(x) \]

\[
\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x^2}
\]

\[
P(x) = \frac{1}{x} \rightarrow x \to 0
\]

\[
f(x) = \frac{1}{x^2}
\]

\[ P \text{ and } f \text{ are continuous for } x > 0 \text{ and } x < 0. \]

Since initial condition is given at \( x = 1 \), we take \( x > 0 \), i.e. \( I = (0, \infty) \).

Find integrating factor: \[ e^{\int P(x) \, dx} \]

\[ e^{\int \frac{1}{x} \, dx} = e^{\ln|x|} = x \text{ for } x > 0. \]

Multiply \[ x \frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x^2} \] by integrating factor \( x \).

\[ \Rightarrow \quad x \frac{dy}{dx} + y = \frac{1}{x} \]

\[
\int \frac{d}{dx} (x \cdot y) \, dx = \int \frac{1}{x} \, dx \quad \text{integrate}
\]

\[ x \cdot y = \ln|x| + C \quad \text{solve for } y. \]
\[ y = \frac{\ln |x|}{x} + \frac{C}{x} \]  

**general solution.**

\[ y'(1) = \frac{y}{2} \]

\[ 2 = \frac{\ln |1|}{1} + \frac{C}{1} \Rightarrow C = 2 \]

So

\[ y(x) = \frac{\ln |x|}{x} + \frac{2}{x} \quad \text{on } (0, \infty) \]

5) \( t = \) time (in minutes) since pumping has began:

\[ A(t) = \text{amount (in lbs) of salt in solution at time } t. \]

\[ \frac{dA}{dt} = \text{rate of change of salt in tank with respect to time (in lbs/min)} \]

\[ \frac{dA}{dt} = \left( \text{Rate at which salt enters tank} \right) - \left( \text{Rate at which salt leaves tank} \right) \]

\[ \text{Rin} \]

\[ \text{Rout} \]

**Rin:** Since pure water is pumped in,

\[ \text{Rin} = 0 \]
Solution leaves tank at a rate of 3gal/min.

\[ \text{Rout} = \frac{3A}{200 + (\text{Rin} - \text{Rout})} \]

\[ \text{fluid is pumped in and out at same rate} \Rightarrow \text{Rin} - \text{Rout} = 0 \]

\[ \text{Rout} = \frac{3A}{200} \text{ lbs/min} \]

Therefore: \[ \frac{\text{d}A}{\text{d}t} = \text{Rin} - \text{Rout} = 0 - \frac{3A}{200} \]

\[ \frac{\text{d}A}{A} = -\frac{3}{200} \text{ dt} \]

\[ \int \frac{\text{d}A}{A} = \int -\frac{3}{200} \text{ dt} \]

\[ \ln |A| = -\frac{3}{200} t \]

\[ A = A_0 e^{-\frac{3}{200} t} \]

\[ t = 0 \Rightarrow A = A_0 = 50 \]
\[ \frac{dy}{dx} = 1 - y^2 = (1-y)(1+y) \]

a) \( (1-y)(1+y) = 0 \)

\[ y = 1, \quad y = -1 \] Critical points

Phase portrait

b) \( \frac{dy}{dx} < 0 \)

1 is asymptotically stable

-1 is unstable.

c) Equilibrium solutions: \( y(x) = -1, \quad y(x) = 1 \)
7) \[ x^2 y'' + 5xy' - 12y = 0 \]

Write in standard form: \[ y'' + \frac{5}{x} y' - \frac{12}{x^2} y = 0 \]

\[ p(x) = \frac{5}{x} \]

\[ y_1 = x^2 \]

\[ y_2 (x) = y_1 \int \frac{e^{-\int \frac{5}{x} dx}}{y_1^2} dx \]

\[ = x^2 \int \frac{e^{-\int \frac{5}{x} dx}}{x^4} dx \]

\[ = -\frac{1}{3} x^{-6} \]

\[ y_1 = x^2 \quad y_2 = -\frac{1}{3} x^{-6} \]

So general solution \[ y = c_1 y_1 + c_2 y_2 \]

\[ y = c_1 x^2 - \frac{c_2}{3} x^{-6} \]

OR
\[ x^2 y'' + 5xy' - 12y = 0 \]

\[
\begin{cases}
y_2 = u(x) y_1(x) \rightarrow y_2 = u(x) x^2 \\
y_0' = u' x^2 + u \cdot 2x \\
y_2'' = u'' x^2 + 4u' x + 2u
\end{cases}
\]

Substitute into eq.

\[ x^2 (u'' x^2 + u u' x + 2u) + 5x (u' x^2 + u \cdot 2x) - 12u x^2 = 0 \]

\[ x^4 u'' + 4x^3 u' + 2x^2 u + 5x^2 u' + 10x^2 u - 12x^2 u = 0 \]

\[ x^4 u'' + 9x^3 u' + 0 = 0 \]

\[ u' = v \quad u'' = v' \]

\[ x^4 v' + 9x^3 v = 0 \]

\[ \frac{dv}{dx} + \frac{9}{x} v = 0 \quad \text{(separable)} \]

\[ \frac{dv}{dx} = -\frac{9}{x} v \rightarrow \left( \frac{dv}{v} = -\frac{9}{x} \right) \]

\[ \ln |v| = -9 \ln |x| + C \quad v = C x^{-9} \]

\[ \ln |x^3 v| = C \quad v = x^3 \]

\[ e^c = v x^9 \]

\[ v = u' \]
\[ u' = C x^{-9} \]

\[ \int du = \int C x^{-9} \, dx \]

\[ u = -\frac{C x^{-8}}{8} + C_0 \]

\[ y_2(x) = u \cdot x^2 = -\frac{C x^{-6}}{8} + C_0 x^2 \]

general solution:

\[ y(x) = C_1 x^2 + C_2 \left( -\frac{C x^{-6}}{8} + C_0 x^2 \right) \]

\[ y(x) = C_1 x^2 - \frac{C_2}{3} x^{-6} \]

8) Look at Answer Key of Quiz 2

Problem #2