1. **Autonomous first-order DEs:**

Step 1. Find critical points and equilibrium solutions:

A real number \( c \) is called a critical point (or equilibrium point or stationary point) of an autonomous ODE \( \frac{dy}{dx} = f(y) \) if \( f(c) = 0 \).

If \( c \) is a critical point of \( \frac{dy}{dx} = f(y) \) then \( y(x) = c \) is a constant solution of \( \frac{dy}{dx} = f(y) \).

A constant solution to \( \frac{dy}{dx} = f(y) \) is called an equilibrium solution.

**Example:** ON BOARD

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Step 2. Determine the phase portrait:

Critical points are used to determine the (one-dimensional) phase portrait or phase line of an autonomous ODE.

**Example:** ON BOARD

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Step 3. Sketch Solution Curves:

Phase portraits (or phase lines) can be used to sketch solution curves of autonomous ODEs.
Since \( f = f(y) \) (i.e., \( f \) is independent of \( x \)) then \( f \) is defined for all meaningful \( x \) values, i.e., \(-\infty < x < \infty\) or \(0 \leq x < \infty\).

Suppose \( y' = f(y) \) has exactly two distinct critical points \( c_1 \) and \( c_2 \), where \( c_1 < c_2 \).

The equilibrium solutions are \( y(x) = c_1 \) and \( y(x) = c_2 \) whose graphs are horizontal lines that partition \( R \) into three subregions \( R_1, R_2 \) and \( R_3 \):

\[
R_1 = \{(x,y) \in R : y < c_1\}
\]
\[
R_2 = \{(x,y) \in R : c_1 < y < c_2\}
\]
\[
R_3 = \{(x,y) \in R : y > c_2\}
\]

Example: ON BOARD

The following properties of nonconstant solutions \( y(x) \) of \( y' = f(y) \) can be shown:

1. If \((x_0, y_0)\) is in subregion \( R_i \) and \( y(x) \) passes through this point, then \( y(x) \) remains in subregion \( R_i \) for all \( x \). That is, the graph of a nonconstant solution cannot cross the graph of an equilibrium solution.

2. \( f(y) \) cannot change signs in a subregion \( R_i \) (since it is continuous), i.e.,

\[
\begin{align*}
&f(y(x)) > 0 \text{ for all } x \in R_i \text{ or } \\
&f(y(x)) < 0 \text{ for all } x \in R_i, (i = 1, 2, 3).
\end{align*}
\]

Therefore, since \( \frac{dy}{dx} = f(y(x)) \), a nonconstant solution \( y(x) \) is strictly monotonic in subregion \( R_i \). That is, a nonconstant solution \( y(x) \) is either strictly increasing in \( R_i \) or strictly decreasing in \( R_i \). Specifically, \( y(x) \) cannot oscillate or have local maxima or minima.
3. The graph of $y(x)$ approaches the graph of one or more equilibrium solutions.

(a) If $y(x)$ is bounded above by critical point $c_1$ (i.e., $y(x)$ is in $R_1$), then the graph of $y(x)$ approaches the graph of the equilibrium solution $y(x) = c_1$ either as $x \to \infty$ or as $x \to -\infty$.

(b) If $y(x)$ is bounded below by critical point $c_2$ (i.e., $y(x)$ is in $R_3$), then the graph of $y(x)$ approaches the graph of the equilibrium solution $y(x) = c_2$ either as $x \to \infty$ or as $x \to -\infty$.

(c) If $y(x)$ is bounded above and below by critical point $c_2$ and $c_1$, respectively, (i.e., $y(x)$ is in $R_2$), then the graph of $y(x)$ approaches the graphs of the equilibrium solutions $y(x) = c_2$ and $y(x) = c_1$, one as $x \to \infty$ and the other as $x \to -\infty$.

Example: ON BOARD
2. **Classification of Critical Points:** A critical point \( c \) is called:

- **asymptotically stable** if all solutions \( y(x) \) of \( \frac{dy}{dx} = f(y) \) that start from an initial point \( (x_0, y_0) \) sufficiently close to \( y = c \) approach \( y = c \) as \( x \to \infty \), i.e., if \( \lim_{x \to \infty} y(x) = c \) for any solution \( y(x) \) that starts sufficiently close to \( y = c \). In this case, \( c \) is called an attractor.

![Asymptotically Stable](image)

- **unstable (or a repeller)** if all solutions \( y(x) \) of \( \frac{dy}{dx} = f(y) \) that start from an initial point \( (x_0, y_0) \) sufficiently close to \( y = c \) move away from \( y = c \) as \( x \) increases.

![Unstable](image)

- **semi-stable** if some solutions of \( \frac{dy}{dx} = f(y) \) that start sufficiently close to \( y = c \) approach \( y = c \) and some move away from \( y = c \) as \( x \) increases.

![Semi-stable](image)
3. Find and classify all the equilibrium solutions to the following differential equation.
\[ y' = (y^2 - 4)(y + 1)^2 \]

\[ f(y) = 0 \Rightarrow (y^2 - 4)(y + 1)^2 = 0 \Rightarrow \]
\[ y^2 - 4 = 0 \quad y = \pm 2 \]
\[ (y + 1)^2 = 0 \quad y = -1 \]

\[ \uparrow \quad \downarrow \]
\[ 2 \quad < \text{unstable} \]
\[ \downarrow \quad \downarrow \]
\[ -1 \quad < \text{semi-stable} \]
\[ \uparrow \quad \uparrow \]
\[ -2 \quad < \text{stable} \]

Phase portrait

4. Consider the dynamical system \( \frac{dy}{dt} = 3(y - 2)(y - 1)(y + 1) \). Find and classify all the equilibrium solutions. Sketch a phase portrait and possible solution curves.
5. Consider the differential equation \( \frac{dz}{dt} = -3(z - 5)(z - 10) \). Find and classify all the equilibrium solutions. Sketch a phase portrait and possible solution curves.

\[ f(z) = -3(z - 5)(z - 10) = 0 \]

\[ z = 5, \quad z = 10 \]

\( 10 \leftrightarrow \text{stable} \)

\( 5 \leftrightarrow \text{unstable} \)