THE SU(3) CASSON INVARIANT FOR INTEGRAL HOMOLOGY 3-SPHERES

HANS U. BODEN AND CHRISTOPHER M. HERALD

Abstract. We derive a gauge theoretic invariant of integral homology 3-spheres which counts gauge orbits of irreducible, perturbed flat SU(3) connections with sign given by spectral flow. To compensate for the dependence of this sum on perturbations, the invariant includes contributions from the reducible, perturbed flat orbits. Our formula for the correction term generalizes that given by Walker in his extension of Casson's SU(2) invariant to rational homology 3-spheres.

1. Introduction

Since its introduction in 1985, Casson's invariant [3, 1] has been the focus of intense study. For example, it has been shown that it extends as a $\mathbb{Q}$-valued invariant of oriented 3-manifolds which retains most of the important properties of the original invariant (for details, see [25, 14] and the references contained therein). Its relevance to gauge theory was recognized by C. Taubes, who related it to the Euler characteristic for the instanton homology groups defined by A. Floer [24, 6]. Because Casson's invariant is essentially defined as an algebraic count of the number of conjugacy classes of irreducible representations $\varrho : \pi_1 X \longrightarrow SU(2)$, it is widely believed that there exists a sequence of related invariants $\lambda_{SU(n)}(X)$ which "count" the number of conjugacy classes of irreducible representations $\varrho : \pi_1 X \longrightarrow SU(n)$. One program for realizing these invariants was proposed by S. Cappell, R. Lee, and E. Miller in the research announcement [4].

The present article establishes the existence of such an invariant for the group $SU(3)$ in case X is an integral homology 3-sphere. The main difficulty in defining $\lambda_{SU(n)}(X)$ is that one must first perturb so that the space of irreducible representations is cut

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out transversely, but the resulting (signed) count will depend on the perturbation used. To obtain a well-defined invariant, one must devise a correction term involving only the reducible representations which compensates for this dependence.

In extending Casson's $SU(2)$ invariant to rational homology 3-spheres, K. Walker gave a formula for the correction term using the symplectic geometry and stratified structure of representation varieties associated to a Heegaard splitting of the 3-manifold [25]. Although the situation of $SU(3)$ representations of integral homology 3-spheres is similar to that of $SU(2)$ representations of rational homology 3-spheres (because in both cases there is only one stratum of reducibles to worry about), we adopt a different approach and use instead gauge theory. This means that we view conjugacy classes of representations as gauge orbits of flat connections via holonomy and study the moduli space of solutions to the (perturbed) flatness equation as the critical set of the (perturbed) Chern-Simons functional. The appropriate interpretation of our arguments in the $SU(2)$ case would lead to a gauge-theoretic formula for Walker's invariant (cf. [18, 15]).

We now give a brief outline of the contents of this paper. The rest of this section presents the fundamental notions of 3-manifold $SU(3)$ gauge theory and describes our main result. Section 2 introduces the perturbations and the perturbed flatness equation. Section 3 is devoted to establishing structure theorems for the moduli space of perturbed flat connections and for the parameterized moduli space. It is important to notice that regularity for the parameterized moduli space does not imply that it is smooth; it typically has non-manifold points which we call bifurcation points. These singularities look locally like 'T' intersections.

Section 4 introduces the spectral flow orientation on the moduli spaces. Subsection 4.4 deserves special mention because it contains a comparison of the orientations on different strata of the parameterized moduli space near a bifurcation point. This is a key ingredient in our main result, which is a formula for the $SU(3)$ Casson invariant and the statement that it defines an invariant of integral homology 3-spheres. All of this is explained in section 5 (cf. Theorem 1). The final section contains technical results concerning the existence of perturbations for $SU(3)$ gauge theory.

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1.1. **SU(3) gauge theory.** Suppose X is a closed, oriented 3-manifold and P is a principal $SU(3)$ bundle over X. For topological reasons, P is trivial. Pick a trivialization $P \cong X \times SU(3)$ and denote by $\Omega^p(X; su(3))$ the space of smooth p-forms with values in the adjoint bundle $\operatorname{ad} P \cong X \times su(3)$. Let $\mathcal{A}$ be the space of smooth connections in P; $\mathcal{A}$ is an affine space modeled on $\Omega^1(X; su(3))$. A gauge transformation is a bundle automorphism $g : P \longrightarrow P$, and the group of smooth gauge transformations $\mathcal{G}$ can be identified with $C^\infty(X, SU(3))$. This group acts on $\mathcal{A}$ by $g \cdot A = gAg^{-1} + gdg^{-1}$ with quotient

$$\mathcal{B} = \mathcal{A}/\mathcal{G}.$$

As usual, the gauge group action is not free. Let $\mathcal{A}^*$ denote the subset of irreducible connections, i.e., those with stabilizer $Z(SU(3)) \cong \mathbb{Z}_3$, and set $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$. While $\mathcal{B}$ is singular at gauge orbits with stabilizer different from $\mathbb{Z}_3$, if $\mathcal{A}$ and $\mathcal{G}$ are given the L^2_1 and L^2_2 topologies, respectively, then $\mathcal{B}^*$ inherits the structure of a pre-Banach manifold. For the most part, we will omit the references to the Sobolev completions in this paper because a detailed account of the analysis can be found in [24].

Assume from now on that X is an integral homology 3-sphere unless otherwise specified. Then the stabilizer of any flat connection is isomorphic to $SU(3)$, $U(1)$, or $\mathbb{Z}_3$ (among nonflat connections, there are two other possibilities, $U(1) \times U(1)$ and $S(U(2) \times U(1))$). Let $\mathcal{A}^r$ denote the space of all connections with stabilizer isomorphic to $U(1)$; these are the nonabelian connections which reduce to $S(U(2) \times U(1))$ connections. We adopt the convenient, if not standard, terminology whereby A *reducible* means $A \in \mathcal{A}^r$.

The quotient $\mathcal{B}^r = \mathcal{A}^r/\mathcal{G}$, while a singular stratum of $\mathcal{B}$, is itself a smooth manifold. This may be seen by noticing that $A \in \mathcal{A}^r$ if and only if it is gauge equivalent to a connection whose 1-form takes values in $s(u(2) \times u(1))$, and that this 1-form is unique up to gauge transformations $g \in C^\infty(X, S(U(2) \times U(1)))$. Thus $\mathcal{B}^r \cong \mathcal{A}^*_{S(U(2)\times U(1))} / \mathcal{G}_{S(U(2)\times U(1))}$.

For $A \in \mathcal{A}$, the curvature is the element $F(A) \in \Omega^2(X; su(3))$ defined by

$$F(A) = dA + A \wedge A.$$

Then $A \in \mathcal{A}$ is flat in case $F(A) = 0$, and the moduli space of flat connections is

$$\mathcal{M} = \{A \in \mathcal{A} \mid F(A) = 0\}/\mathcal{G} \ \subset \mathcal{B}.$$

Set $\mathcal{M}^* = \mathcal{M} \cap \mathcal{B}^*$ and $\mathcal{M}^r = \mathcal{M} \cap \mathcal{B}^r$. A well known theorem identifies $\mathcal{M}$ with the space of representations $\varrho : \pi_1 X \longrightarrow SU(3)$ modulo conjugation.

The Chern-Simons functional $CS(A)$ is defined by

$$CS(A) = \frac{1}{8\pi^2} \int_X tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A).$$

There is an isomorphism $\pi_0 \mathcal{G} \cong \mathbb{Z}$ given by $g \mapsto \deg g$ (see Proposition 4.2). If $g \in \mathcal{G}$, then $CS(g \cdot A) = \deg g + CS(A)$, thus CS descends to a map

$$CS : \mathcal{B} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1.$$

Choose an orientation and a Riemannian metric on X. This provides a Hodge star operator $* : \Omega^p(X; su(3)) \longrightarrow \Omega^{3-p}(X; su(3))$ and an L^2 Riemannian metric on $\mathcal{A}$, given by $\langle a, b \rangle_{L^2} = -\int_X tr(a \wedge *b)$. Taking the gradient of CS with respect to this metric, one computes that

$$\nabla \, CS(A) = -\frac{1}{4\pi^2} * F(A),$$

and hence the set of critical points of CS, modulo $\mathcal{G}$, is exactly the moduli space of flat connections $\mathcal{M}$.

The linearization of the flatness equation $*F(A) = 0$ is given by the operator $*d_A : \Omega^1(X; su(3)) \longrightarrow \Omega^1(X; su(3))$. As in [24], we extend this to the self-adjoint, elliptic operator

$$K_A : \Omega^0(X; su(3)) \oplus \Omega^1(X; su(3)) \longrightarrow \Omega^0(X; su(3)) \oplus \Omega^1(X; su(3))$$
$$K_A(\xi, a) = (d_A^* a, d_A \xi + *d_A a).$$

Notice that $\ker K_A = \mathcal{H}^0_A(X; su(3)) \oplus \mathcal{H}^1_A(X; su(3))$, the space of d_A-harmonic (0+1)-forms.

For X any closed 3-manifold, the moduli space of flat $SU(3)$ connections $\mathcal{M}$ is compact and has expected dimension zero since K_A is self-adjoint. Achieving transversality requires the use of perturbations, and we employ the same techniques here that were successful in the $SU(2)$ setting [24, 9, 10].

We define a class of admissible perturbation functions in Section 2 by which to vary the Chern-Simons functional. The construction of an admissible function h involves taking a sum of invariant functions applied to the holonomy around a collection of loops (integrated over normal disks of tubular neighborhoods of the loops). The perturbed Chern-Simons functional is then $CS_h(A) = CS(A) + h(A)$, and a connection is called h-perturbed flat if it is a critical point of CS_h. We show in Section 3 that it is possible to choose an admissible function h such that $\mathcal{M}^*_h$ and $\mathcal{M}^r_h$ are compact 0-dimensional submanifolds of $\mathcal{B}^*$ and $\mathcal{B}^r$ consisting of orbits that meet a cohomological regularity condition.

1.2. **Main result.** We begin by recalling from [24] the gauge-theoretic definition of Casson's invariant $\lambda(X)$ in case X is an integral homology 3-sphere. First, choose a small perturbation h so that the perturbed flat $SU(2)$ moduli space is a compact, smooth, oriented 0-manifold. Then the number of irreducible, perturbed flat connections counted with sign is seen to be independent of the choice of perturbation h. This follows from the classification of 1-manifolds once it is verified that for generic, one-parameter families of perturbations, the irreducible part of the parameterized $SU(2)$ moduli space is a smooth cobordism between the two moduli spaces at either end. Taubes identified the resulting invariant as -2 times Casson's invariant, normalized as in [1] (see [13] for an explanation of the minus sign).

In the $SU(3)$ case, for generic one-parameter families $\rho(t) = h_t$ of perturbations, the irreducible part of the parameterized moduli space W_ρ^* is an oriented 1-manifold, but it is not generally compact. The reducible part, W_ρ^r, is a compact 1-manifold, and the union $W_\rho^* \cup W_\rho^r$ is compact but not smooth. The problem is illustrated in Figure 1, where $\rho(t)$ is defined for $t \in [-1, 1]$. The solid curves depict W_ρ^* and the dotted curves W_ρ^r. Because of the noncompact ends of W_ρ^*, the parameterized moduli space subfails to give a smooth cobordism between $\mathcal{M}_{\rho(-1)}^*$ and $\mathcal{M}_{\rho(1)}^*$. Thus the algebraic sum of perturbed flat irreducible orbits is seen to depend on the perturbation in this case.

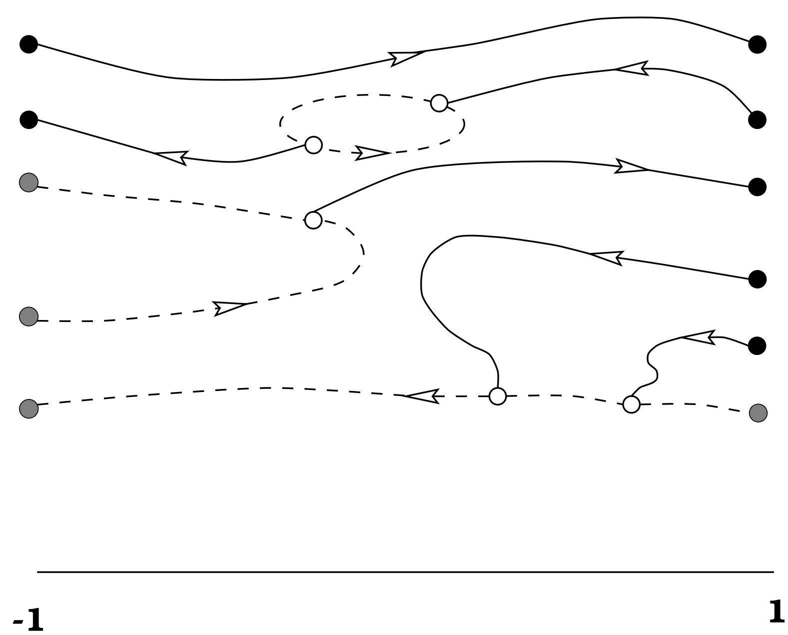

FIGURE 1. The parameterized moduli space $W_\rho^* \cup W_\rho^r$ projecting vertically to $[-1, 1]$.

The compactification $\overline{W_\rho^*}$ is obtained by adding certain reducible orbits, called bifurcation points, to the non-compact ends of W_ρ^*. In Figure 1, the bifurcation points are where the dotted and solid curves meet. To make the invariant independent of h, one needs a correction term which changes, when the perturbation is varied, by the number of bifurcation points on W_ρ^r, counted with sign given by their orientation as boundary points of $\overline{W_\rho^*}$.

The oriented spectral flow along W_ρ^r provides a means to calculate this number, as we now explain. Let $\mathfrak{h} = s(u(2) \times u(1))$ be the Lie subalgebra of $su(3)$ and $\mathfrak{h}^\perp$ its orthogonal complement, which can be identified with $\mathbb{C}^2$. For any reducible connection A, the connection 1-form can be gauge transformed to take values in $\mathfrak{h}$. If A is h-perturbed flat, then $\Omega^1(X; su(3)) = \Omega^1(X; \mathfrak{h}) \oplus \Omega^1(X; \mathfrak{h}^\perp)$ is the splitting of $T_A\mathcal{A}$ into tangent vectors tangent to and normal to the reducible stratum. For generic paths ρ, the bifurcation points are characterized geometrically as those reducible orbits in W_ρ where the kernel of the restriction of $K(A,h)$ to the $\mathfrak{h}^\perp$-valued forms jumps up in dimension. Such a jump occurs each time the deformation complex detects a tangent vector normal to the reducible stratum. Hence, in a neighborhood of the bifurcation point in W_ρ^r, there is a path of eigenvalues of $K(A,h)$ (on $\mathfrak{h}^\perp$-valued forms) crossing zero transversally, and the sign of its first derivative (relative to the orientation on W_ρ^r) coincides with the boundary orientation of the bifurcation point. Note that Stab $A \cong U(1)$ equivariance of $K(A,h)$ forces the eigenvalue to have multiplicity two.

Choosing the product connection θ as a reference point for computing all spectral flows, we obtain:

Theorem 1. *Suppose X is an integral homology 3-sphere. For generic small perturbations h, $\mathcal{M}_h^*$ and $\mathcal{M}_h^r$ are smooth, compact 0-manifolds. Choose representatives A for each orbit $[A] \in \mathcal{M}_h$, and in case $[A] \in \mathcal{M}_h^r$, choose also a flat connection $\widehat{A}$ close to A. Define $\lambda_{SU(3)}(X,h)$ to be equal to*

$$\sum_{[A]\in\mathcal{M}_h^*} (-1)^{Sf(\theta,A)} - \frac{1}{2} \sum_{[A]\in\mathcal{M}_h^r} (-1)^{Sf(\theta,A)} (Sf_{\mathfrak{h}^\perp}(\theta,A) - 4\, CS(\widehat{A}) + 2),$$

where Sf and $Sf_{\mathfrak{h}^\perp}$ refer to the spectral flow of the operator $K(A,h)$ on $su(3)$ and $\mathfrak{h}^\perp$ bundle-valued forms, respectively. Then for h sufficiently small, this quantity is independent of h and the Riemannian metric on X, and gives a well-defined invariant of integral homology 3-spheres.

Remark. This theorem will follow from 3.13 and the results in section 5.

The second sum is our formula for the correction term. Both $Sf_{\mathfrak{h}^\perp}(\theta,A)$ and $CS(\widehat{A})$ depend on the choice of representative A. It is only the difference $Sf_{\mathfrak{h}^\perp}(\theta,A) - 4\, CS(\widehat{A})$

which is well-defined on the gauge orbit $[A]$. The last term in the second sum does not affect the argument that $\lambda_{SU(3)}$ is well-defined; it simply adds a certain multiple of the $SU(2)$ Casson invariant to get a desirable choice of normalization.

As an invariant, $\lambda_{SU(3)}$ is insensitive to the orientation on X. In general, if $\lambda_{SU(3)}(X) \neq 0$, then $\pi_1 X$ admits a non-trivial representation into $SU(2)$ or $SU(3)$. The conjectured rationality of $CS(\widehat{A})$ would of course imply that $\lambda_{SU(3)}(X) \in \mathbb{Q}$ as well.

There are many interesting questions raised by Theorem 1. The most intriguing is what sort of surgery relations (if any) does this new invariant satisfy. A related question:[1] is $\lambda_{SU(3)}$ a finite type invariant [20, 8]? By [19], the Casson-Walker invariant equals 6 times λ_1, the first Ohtsuki invariant [21], so one is especially interested in any relationship between $\lambda_{SU(3)}$ and λ_2, the second Ohtsuki invariant. Positive results would be interesting for two reasons: (i) they would render $\lambda_{SU(3)}$ computable by algebraic means, and (ii) they would clarify what geometric information the finite type invariants carry.

There is, of course, still the problem of defining the generalized Casson $SU(n)$ invariants for $n > 3$. A related problem is to extend $\lambda_{SU(3)}$ to rational homology 3-spheres. In a different direction, one can attempt to define $SU(3)$ Floer theory. We leave these questions to future investigations.

[1] We are grateful to S. Garoufalidis for pointing out the connection here.

2. Perturbations

In this section, we present the functions that will be used to perturb the Chern-Simons functional. After defining the perturbations and characterizing the perturbed flat connections, we derive those properties of the first and second derivative of the perturbation functions which are used later to prove that the critical set of the perturbed Chern-Simons functional satisfies certain transversality conditions.

2.1. **Admissible functions.** This subsection introduces the admissible functions, which are gauge invariant functions $\mathcal{A} \longrightarrow \mathbb{R}$ obtained by applying invariant functions $SU(3) \longrightarrow \mathbb{R}$ to the holonomy around a collection of loops in X. We first describe the construction for a single loop.

Each smoothly embedded based curve $\ell : S^1 \longrightarrow X$ defines a holonomy map

$$hol_\ell : \mathcal{A} \longrightarrow SU(3).$$

We can obtain from this a gauge invariant function $f : \mathcal{A} \longrightarrow \mathbb{R}$ by composing with an invariant function $\tau : SU(3) \longrightarrow \mathbb{R}$. For analytical reasons, it is necessary to mollify this function by integrating against a cut-off function on the 2-disks normal to ℓ as follows.

Let $x = (x_1, x_2)$ be coordinates on D^2, the unit 2-dimensional disk. Fix once and for all a radially symmetric 2-form η on D^2 which vanishes near the boundary and satisfies $\int_{D^2} \eta = 1$. A tubular neighborhood of ℓ is an embedded solid torus $\gamma : S^1 \times D^2 \longrightarrow X$. For each $x \in D^2$, let $hol_\gamma(x, A)$ be the holonomy of A once around the closed curve $\gamma(S^1 \times \{x\})$. For any smooth invariant function $\tau : SU(3) \longrightarrow \mathbb{R}$, define the gauge invariant function $p(\gamma, \tau) : \mathcal{A} \longrightarrow \mathbb{R}$ by

$$p(\gamma, \tau)(A) = \int_{D^2} \tau(hol_\gamma(x, A))\eta(x)dx. \tag{1}$$

Definition 2.1. *Fix $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, a set of embeddings of the solid torus into X. Then an* **admissible function relative to** Γ *is a function $h : \mathcal{A} \longrightarrow \mathbb{R}$ defined by*

$$h(A) = \sum_{i=1}^{n} p(\gamma_i, \tau_i) = \sum_{i=1}^{n} \int_{D^2} \tau_i(hol_{\gamma_i}(x, A))\eta(x)dx.$$

where $\tau_i : SU(3) \longrightarrow \mathbb{R}$ *is an invariant function of the form* $\tau_i = h_i \circ tr$ *for a* C^3 *function* $h_i : \mathbb{C} \longrightarrow \mathbb{R}$*. Given* Γ*, we denote the space of admissible functions by* $\mathcal{F}_\Gamma$ *and note the identification* $\mathcal{F}_\Gamma \cong C^3(\mathbb{C}, \mathbb{R})^{\times n}$ *given by* $h \mapsto (h_1, \dots, h_n)$*. For* $h \in \mathcal{F}_\Gamma$*, define* $\|h\|_{C^3} = \sum_{i=1}^n \|h_i\|_{C^3}$*.*

There is no real loss of generality in considering only the invariant functions of the type used in the previous definition. One can see this by the following result, which we have included for motivation.

Proposition 2.2.

(i) $tr : SU(3) \longrightarrow \mathbb{C}$ *descends to a one-to-one map on conjugacy classes.*

(ii) *Any smooth invariant function* $\tau : SU(3) \rightarrow \mathbb{R}$ *can be written as* $\tau = f \circ tr$ *for some smooth function* $f : \mathbb{C} \longrightarrow \mathbb{R}$*.*

Proof. The characteristic polynomial of $M \in SU(3)$ is given by

$$p_M(\lambda) = \lambda^3 - tr(M)\lambda^2 + \overline{tr(M)}\lambda - 1.$$

Since every matrix in $SU(3)$ is diagonalizable, any two are conjugate if and only if their eigenvalues coincide, and (i) follows.

Part (ii) follows from invariant theory. Consider the case of smooth invariant functions on $U(3)$. Restricting to a maximal torus T^3, these can be viewed as S_3 invariant functions on T^3, where S_3 acts by permutation of the coordinates. The inclusion $T^3 \subset \mathbb{C}^3$ is an equivariant embedding, and a classical result states that the algebra of invariant polynomials $P(\mathbb{C}^n)^{S_n}$ is generated by the elementary, symmetric functions $\sigma_1, \dots, \sigma_n$ (see Chapter 2A, [26]). This, and Theorem 2 of [23], proves (ii), since the σ_i are just the coefficients of the characteristic polynomial, which, for $M \in SU(3)$, are given by $tr(M)$ and $\overline{tr(M)}$. □

2.2. **Perturbed flat connections.** In this subsection, we introduce the perturbed flatness equation and the deformation complex of the perturbed flat moduli space. Suppose that $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ is a set of embeddings of the solid torus into X. All the admissible functions in this section are to be regarded as admissible relative to Γ.

Pick a Riemannian metric on X and let $* : \Omega^p(X; su(3)) \to \Omega^{3-p}(X; su(3))$ be the Hodge star operator. This defines an L^2 inner product on bundle-valued p-forms by

$$\langle \alpha, \beta \rangle_{L^2} = -\int_X tr(\alpha \wedge *\beta)$$

and induces an L^2 metric on $\mathcal{A}$. For any admissible function $h : \mathcal{A} \longrightarrow \mathbb{R}$, let ∇h be the gradient of h with respect to the L^2 metric and define

$$\zeta_h : \mathcal{A} \longrightarrow \Omega^1(X; su(3))$$

by $\zeta_h(A) = *F(A) - 4\pi^2 \nabla h(A)$. Notice that $\zeta_h(A)$ is just $-4\pi^2$ times the gradient of the function from $\mathcal{A}$ to $\mathbb{R}$ given by $A \mapsto CS(A) + h(A)$.

Definition 2.3. *Suppose h is an admissible function. Then $A \in \mathcal{A}$ is called h-* **perturbed flat** *if it satisfies*

$$*F(A) - 4\pi^2 \nabla h(A) = 0.$$

The **perturbed flat moduli space** *is the set of gauge orbits of perturbed flat connections, i.e.,*

$$\mathcal{M}_h = \zeta_h^{-1}(0)/\mathcal{G}.$$

Set $\mathcal{M}_h^* = \mathcal{M}_h \cap \mathcal{B}^*$ and $\mathcal{M}_h^r = \mathcal{M}_h \cap \mathcal{B}^r$.

Definition 2.4. *Suppose $\rho(t)$, $-1 \leq t \leq 1$, is a one-parameter family of admissible functions. Then the* **parameterized moduli space** *is defined as the quotient*

$$W_\rho = \{(A, t) \in \mathcal{A} \times [-1, 1] \mid \zeta_{\rho(t)}(A) = 0\}/\mathcal{G} \subset \mathcal{B} \times [-1, 1],$$

with slice at $t \in [-1, 1]$ given by $\mathcal{M}_{\rho(t)} \times \{t\} = W_\rho \cap (\mathcal{B} \times \{t\})$. Set $W_\rho^* = W_\rho \cap (\mathcal{B}^* \times [-1, 1])$ and $W_\rho^r = W_\rho \cap (\mathcal{B}^r \times [-1, 1])$.

Since X is an integral homology 3-sphere, any reducible *flat* connection can be regarded as an irreducible, flat $SU(2)$ connection. This is no longer true for perturbed flat reducible connections because they typically have holonomy in a subgroup conjugate to $S(U(2) \times U(1))$ and do not reduce any further.

The linearization of ζ_h is given by

$$*d_{A,h} = *d_A - 4\pi^2 \operatorname{Hess} h(A) : \Omega^1(X; su(3)) \longrightarrow \Omega^1(X; su(3)).$$

This motivates the final definition of this subsection.

Definition 2.5. *Suppose that h is an admissible function and that A is h-perturbed flat. The* **deformation complex** *is the elliptic Fredholm complex*

$$\Omega^0(X; su(3)) \xrightarrow{d_A} \Omega^1(X; su(3)) \xrightarrow{*d_{A,h}} \Omega^1(X; su(3)) \xrightarrow{d_A^*} \Omega^0(X; su(3)), \tag{2}$$

where d_A^* is the L^2-adjoint of d_A. The first two cohomology groups of this complex are $H^0_A(X; su(3)) = \ker d_A$ and $H^1_{A,h}(X; su(3)) = \ker *d_{A,h}/\operatorname{im} d_A$. Notice that this is a self-adjoint complex, and so cohomological groups of complementary dimensions are identified.

Of course, if $h = 0$, then (2) is just the twisted de Rham complex with the second half rewritten using duality. We will represent $H^0_A(X; su(3))$ and $H^1_{A,h}(X; su(3))$ by the spaces $\mathcal{H}^0_A(X; su(3))$ and $\mathcal{H}^1_{A,h}(X; su(3))$ of harmonic forms, where a 1-form a is *harmonic* if $d_A a = 0$ and $*d_{A,h}(a) = 0$. Geometrically, the former cohomology group is the Lie algebra of $\operatorname{Stab}(A)$, while the latter is the kernel of the linearized perturbed flatness equation restricted to the tangent space to the slice of the gauge group action.

Given a complex line $V \subset \mathbb{C}^3$, we can decompose $\mathbb{C}^3$ into V and $V^\perp$. This gives an identification, typically different from the standard one, between $\mathbb{C}^3$ and $\mathbb{C} \oplus \mathbb{C}^2$. This engenders a corresponding decomposition of the Lie algebra as $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$, isomorphic (as a vector space) to $s(u(2) \times u(1)) \oplus \mathbb{C}^2$. For example, for the standard decomposition,

$$\mathfrak{h} = \left\{ \begin{pmatrix} i(a+b) & c+id & 0 \\ -c+id & i(a-b) & 0 \\ 0 & 0 & -2ia \end{pmatrix} \right\} \text{ and } \mathfrak{h}^\perp = \left\{ \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix} \right\}.$$

In general, $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are given by conjugating the above subspaces.

If A is a connection in the bundle $P = X \times SU(3)$ and $\operatorname{Stab} A \cong U(1)$, then the action of $\operatorname{Stab} A$ on the canonical $\mathbb{C}^3$ bundle $E \longrightarrow X$ decomposes each fiber of $\operatorname{ad} P$ in a similar manner. We shall use the notation $\mathfrak{h}$ and $\mathfrak{h}^\perp$ without indicating the actual dependence of the splitting of $\operatorname{ad} P$ on the subgroup $\operatorname{Stab} A \subset \mathcal{G}$; one can always gauge transform A into $\mathcal{A}_{S(U(2)\times U(1))}$ and then $\operatorname{Stab} A$ would just give the standard decomposition.

For $A \in \mathcal{A}^r$, we decompose 1-forms in a similar manner, and $\Omega^1(X; su(3)) = \Omega^1(X;\mathfrak{h}) \oplus \Omega^1(X;\mathfrak{h}^\perp)$ is a geometric splitting of the tangent space $T_A\mathcal{A}$ into vectors tangent to the reducible stratum $\mathcal{A}^r$ and vectors normal to that stratum. If A is h-perturbed flat, this leads to a splitting of the cohomology groups as

$$\mathcal{H}^*_{A,h}(X; su(3)) = \mathcal{H}^*_{A,h}(X;\mathfrak{h}) \oplus \mathcal{H}^*_{A,h}(X;\mathfrak{h}^\perp).$$

For convenience, set $\Omega^{0+1}(X; su(3)) = \Omega^0(X; su(3)) \oplus \Omega^1(X; su(3))$. We can fold the deformation complex (2) up into a single operator

$$K(A, h) : \Omega^{0+1}(X; su(3)) \longrightarrow \Omega^{0+1}(X; su(3))$$

by setting, for $(\xi, a) \in \Omega^0(X; su(3)) \oplus \Omega^1(X; su(3))$,

$$K(A, h)(\xi, a) = (d_A^* a, d_A\xi + *d_{A,h}(a)).$$

Notice that $K(A, h)$ is a self-adjoint elliptic operator (with appropriate Sobolev norms on the domain and range). When A is reducible, the operator $K(A, h)$ respects the decomposition of $\Omega^1(X; su(3))$ described above. In particular, in Sections 4 and 5, we use this to split the spectral flow of $K(A, h)$.

2.3. **The calculus of admissible functions.** In this subsection, we describe the first and second derivatives of functions $f : \mathcal{A} \longrightarrow \mathbb{R}$ obtained by composing the holonomy around a loop with an invariant function $\tau : SU(3) \longrightarrow \mathbb{R}$ as in eqn. (1).

For such functions, these computations can all be performed on the pullback bundles over S^1. Hence, throughout this section, $\mathcal{A}$ denotes the space of connections on the bundle $P = S^1 \times SU(3)$. Parameterize the circle by $f : [0, 1] \to S^1$, $f(u) = e^{2\pi i u}$. For $A \in \mathcal{A}$, let $hol(A) \in SU(3)$ be the holonomy once around the circle in a counterclockwise direction, based at $1 = f(0)$.

The derivatives of $hol(A)$ may be computed as follows. For $A \in \mathcal{A}$, parallel translation by A defines a trivialization of the pullback bundle $f^*(\operatorname{ad} P)$, which identifies tangent vectors in $T_A\mathcal{A}$ with functions $a : [0,1] \to su(3)$.

Proposition 2.6. *Suppose $A \in \mathcal{A}$ and $a, b \in T_A\mathcal{A}$. Then*

(i) $\frac{d}{dt}\, hol(A + ta)\big|_{t=0} = hol(A) \int_0^1 a(\nu) d\nu$,

(ii) $\frac{\partial^2}{\partial s \partial t}\, hol(A + ta + sb)\Big|_{(0,0)} = hol(A) \int_0^1 \int_0^\nu (a(\nu)b(\mu) + b(\nu)a(\mu)) d\mu d\nu.$

Proof. We prove (ii) and leave (i) as an exercise for the reader.

Let $P(s,t;u) \in SU(3)$ denote the parallel translation with respect to the fixed trivialization from 0 to u along the interval by the connection $A + sa + tb$. Then $P(s,t;u)$ satisfies the differential equation

$$\tfrac{\partial}{\partial u} P(s,t;u) + (sa(u) + tb(u))P(s,t;u) = 0. \tag{3}$$

Applying $\frac{\partial^2}{\partial s \partial t}$ to (3) at $(s,t) = (0,0)$, we obtain

$$\tfrac{\partial}{\partial u}\left(\tfrac{\partial^2}{\partial s\partial t} P(s,t;u)\Big|_{(0,0)}\right) + a(u)\tfrac{\partial}{\partial t} P(0,t;u)\big|_{t=0} + b(u)\ \tfrac{\partial}{\partial s} P(s,0;u)\big|_{s=0} = 0.$$

Integrating with respect to u and commuting mixed partials gives

$$\tfrac{\partial^2}{\partial s\partial t} P(s,t;u)\Big|_{(0,0)} = -\int_0^u \left(a(\nu)\ \tfrac{\partial}{\partial t} P(0,t;\nu)\big|_{t=0} + b(\nu)\ \tfrac{\partial}{\partial s} P(s,0;\nu)\big|_{s=0}\right) d\nu.$$

The equations $\frac{\partial}{\partial t} P(0,t;\nu)\big|_{t=0} = -\int_0^\nu b(\mu) d\mu$ and $\frac{\partial}{\partial s} P(s,0;\nu)\big|_{s=0} = -\int_0^\nu a(\mu) d\mu$ can be obtained from (3) in a similar manner, using that $P(0,0;u)$ is the identity. Substituting each of these into the equation above and evaluating at $u = 1$ gives the desired result since $hol(A + sa + tb) = hol(A)P(s,t;1)$. $\square$

This proposition allows us to compute the first and second derivatives of any function $f : \mathcal{A} \longrightarrow \mathbb{R}$ of the form $f = \tau \circ hol$, where $\tau : SU(3) \longrightarrow \mathbb{R}$ is a smooth invariant function. An important example is when τ is either the real or imaginary part of $tr : SU(3) \longrightarrow \mathbb{C}$.

Corollary 2.7. *The first and second derivatives of the trace of holonomy are given by:*

(i) $\frac{d}{dt}\, tr(hol(A+ta))\big|_{t=0} = \int_0^1 tr\,(hol(A)a(\mu))\, d\mu$,

(ii) $\frac{\partial^2}{\partial s \partial t}\, tr(hol(A+sa+tb))\Big|_{(0,0)} = \int_0^1 \int_0^\nu tr\{hol(A)(a(\nu)b(\mu)+b(\nu)a(\mu))\} d\mu d\nu.$

Remark. Proposition 2.6 and Corollary 2.7 remain valid for $SU(n)$, $n > 3$.

In Section 3, we shall show that for a suitable choice of Γ, regularity of $\mathcal{M}_h$ is a generic condition for $h \in \mathcal{F}_\Gamma$ near zero, and similarly for regularity of W_ρ for $\rho \in C^1([-1,1], \mathcal{F}_\Gamma)$. The following proposition provides useful bounds on the derivatives of admissible functions.

Proposition 2.8. (i) *Fix $\gamma : S^1 \times D^2 \longrightarrow X$ an embedding of the solid torus and let τ_1, τ_2 be the real and imaginary parts of trace on $SU(3)$. Then there exists a constant C_1 depending on γ such that*

$$|D^n p(\gamma, \tau_j)(A)(a_1, \ldots, a_n)| \leq C_1 \prod_{i=1}^{n} \|a_i\|_{L^2_1}$$

for all $A \in \mathcal{A}$ and for $j = 1, 2$.

(ii) *Fix Γ a collection of embedded solid tori. Then there exists a constant C_2 depending on Γ such that the inequalities hold for all $h \in \mathcal{F}_\Gamma$ and all $A \in \mathcal{A}$*

$$\begin{aligned}
|Dh(A)(a_1)| &\leq C_2\, \|h\|_{C^3} \cdot \|a_1\|_{L^2_1}, \\
|D^2 h(A)(a_1, a_2)| &\leq C_2\, \|h\|_{C^3} \cdot \|a_1\|_{L^2} \cdot \|a_2\|_{L^2}, \\
|D^3 h(A)(a_1, a_2, a_3)| &\leq C_2\, \|h\|_{C^3} \cdot \|a_1\|_{L^2_1} \cdot \|a_2\|_{L^2} \cdot \|a_3\|_{L^2}, \\
\|\nabla h(A)\|_{L^2_1} &\leq C_2\, \|h\|_{C^3}.
\end{aligned}$$

Proof. See [24], Section 8a. □

The last proposition of this section allows one to patch together the local regularity arguments to give global results in subsection 3.1.

Proposition 2.9. *If $C \subset \mathcal{F}$ is compact, then $\bigcup_{h \in C} \mathcal{M}_h$ is also compact.*

Proof. See Lemma 8.3 in [24]. □

3. Transversality

The goal of this section is to establish various structure theorems for the perturbed flat moduli space $\mathcal{M}_h$ and for the parameterized moduli space W_ρ for generic $h \in \mathcal{F}$ and generic $\rho \in C^1([-1,1], \mathcal{F})$. Before doing this, we must fix a collection Γ of solid tori so that the resulting space of perturbations $\mathcal{F}_\Gamma$ is general enough for these transversality results to hold.

The first subsection contains a formulation of the necessary conditions on Γ and a result which implies that we can always choose Γ to satisfy these conditions in a neighborhood of $\mathcal{M}$ in $\mathcal{B} \times \mathcal{F}_\Gamma$. In the second subsection, we proceed with the transversality results for $\mathcal{M}_h$ and W_ρ.

3.1. **Abundance of admissible functions.**

For any $A \in \mathcal{A}$, define

$$\mathcal{K}_A = \ker d_A^* \cap \Omega^1(X; su(3))$$

and denote by $\Pi_A : \Omega^1(X; su(3)) \longrightarrow \mathcal{K}_A$ the L^2 orthogonal projection. The slice through A to the gauge action is the affine subspace

$$X_A = \{A + a \mid a \in \mathcal{K}_A\} \subset \mathcal{A}.$$

A small neighborhood of A in X_A, divided by the stabilizer of A, gives a local model for $\mathcal{B}$ near $[A]$.

The first proposition reduces the study of the local structure of the moduli space to a Fredholm problem.

Proposition 3.1. *Given a perturbed flat connection, there is a neighborhood $U \subset X_A$ of A such that $A + a \in U$ implies that $\zeta_h(A+a) = 0$ if and only if $\Pi_A \zeta_h(A+a)$.*

Proof. See Lemma 12.1.2 of [17] and Lemmas 28 and 29 of [9]. □

Definition 3.2. *Suppose A is a reducible h-perturbed flat connection and denote by $\mathrm{Herm}\,\mathcal{H}^1_{A,h}(X; \mathfrak{h}^\perp)$ the set of $\mathrm{Stab}(A) \cong U(1)$ invariant symmetric (hence Hermitian) bilinear forms on $\mathcal{H}^1_{A,h}(X; \mathfrak{h}^\perp)$.*

Definition 3.3. *A collection* Γ *of embedded solid tori in* X *is called* **abundant** *for* (A, h)*, where* $h \in \mathcal{F}_\Gamma$ *and* $A \in \mathcal{A}^* \cup \mathcal{A}^r$ *is h-perturbed flat, in case there exists a finite subset* $\{f_1, \ldots, f_m\} \subset \mathcal{F}_\Gamma$ *of admissible functions such that:*

(i) *If* $A \in \mathcal{A}^*$*, then the map from* $\mathbb{R}^m$ *to* $\operatorname{Hom}(\mathcal{H}^1_{A,h}(X; su(3)), \mathbb{R})$ *given by* $(x_1, \ldots, x_m) \mapsto \sum_{i=1}^m x_i Df_i(A)$ *is surjective.*

(ii) *If* $A \in \mathcal{A}^r$*, then the map from* $\mathbb{R}^m$ *to* $\operatorname{Hom}(\mathcal{H}^1_{A,h}(X; \mathfrak{h}), \mathbb{R}) \oplus \operatorname{Herm} \mathcal{H}^1_{A,h}(X; \mathfrak{h}^\perp)$ *given by* $(x_1, \ldots, x_m) \mapsto (\sum_{i=1}^m x_i Df_i(A), \sum_{i=1}^m x_i \operatorname{Hess} f_i(A))$ *is surjective.*

Because abundance is a gauge invariant concept, it makes sense to say that Γ *is abundant for* $([A], h)$*. When* $h = 0$*, we say that* Γ *is abundant for* A *or* $[A]$*.*

If Γ is abundant for (A, h) and $\Gamma \subset \Gamma'$, then of course Γ' is also abundant for (A, h). The next proposition is the principal result of this subsection; it shows that there exists a collection Γ which is abundant for all nontrivial perturbed flat connections in a neighborhood of the flat moduli space. This is a global result and its proof will occupy the remainder of the subsection. The statement of the proposition is divided into three parts, which can be viewed as the pointwise, local, and global versions of the same result.

Proposition 3.4. (i) *If* $A \in \mathcal{A}$ *is a nontrivial flat connection, then there exists a finite collection* Γ *which is abundant for* A*. In case* A *is reducible,* Γ *and the subset* $\{f_1, \ldots, f_m\}$ *from Definition 3.3 can be chosen so that for some* k*,*

(a) $\{Df_1(A), \ldots, Df_k(A)\}$ *spans* $\operatorname{Hom}(\mathcal{H}^1_A(X; \mathfrak{h}), \mathbb{R})$

(b) $\{\operatorname{Hess} f_{k+1}(A), \ldots, \operatorname{Hess} f_m(A)\}$ *spans* $\operatorname{Herm} \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$

(c) $Df_j(A) = 0$ *for* $j = k+1, \ldots, m$*.*

(ii) *If* $A \in \mathcal{A}$ *is a nontrivial flat connection and* Γ *is abundant for* A *and is chosen as in (i), then there exists an open neighborhood* $U \times V \subset \mathcal{B} \times \mathcal{F}_\Gamma$ *of* $([A], 0)$ *such that* Γ *is abundant for all* $([A'], h) \in U \times V$ *with* $\zeta_h(A') = 0$*.*

(iii) *There exist a finite collection* Γ *and an open neighborhood* $U \times V \subset \mathcal{B} \times \mathcal{F}_\Gamma$ *of* $\mathcal{M} \setminus [\theta]$ *such that* Γ *is abundant for all* $([A], h) \in U \times V$ *with* $\zeta_h(A) = 0$*.*

Proof. Part (i) follows from Corollary 6.7 and Proposition 6.8, as we now explain. First, assume A is irreducible. Replace all loops ℓ coming from 6.7 (ii) by tubular neighborhoods γ. Next, by shrinking the tubular neighborhoods, if necessary, we can approximate functions $f : \mathcal{A} \longrightarrow \mathbb{C}$ of the form $f(A) = tr(hol_\ell(A))$ arbitrarily closely by the complex-valued functions $p(\gamma, tr)(A)$ defined as in equation (1). In case A is reducible, apply the same procedure to obtain real-valued functions $p(\gamma, tr_{\mathbb{R}})(A)$ from the real part of $tr(hol_\ell(A))$ for the loops in 6.7 (i). This proves (i) for A irreducible as well as part (a) for A reducible.

To finish off part (i) in case A is reducible, thicken the loops obtained from an application of Proposition 6.8. This provides a collection of functions with $Dp(\gamma, tr)(A) = 0$ whose Hessians span Herm $\mathcal{H}^1_{A,h}(X; \mathfrak{h}^\perp)$. This proves (b) and (c) and completes the proof of part (i).

Part (ii) says that abundance is an open condition around flat connections in $\mathcal{A} \times \mathcal{F}_\Gamma$ and requires several estimates, contained in Lemmas 3.5 and 3.6. Before presenting those arguments, we explain how (iii) follows from (i) and (ii).

By (i) and (ii), for any nontrivial flat connection A, we have a collection Γ which is abundant for all perturbed flat orbits $([A'], h)$ in a neighborhood $U' \times V' \subset \mathcal{B} \times \mathcal{F}_\Gamma$ of $([A], 0)$. Applying this for each $[A] \in \mathcal{M} \setminus [\theta]$ and using compactness, we obtain a finite subcover $U'_1, \ldots, U'_l$ and corresponding collections $\Gamma_1, \ldots, \Gamma_l$. Set $\Gamma = \bigcup_{i=1}^l \Gamma_i$. Part (iii) follows by applying (ii) once again to A and the collection Γ to obtain an open neighborhood $U \times V \subset \mathcal{B} \times \mathcal{F}_\Gamma$ of $([A], 0)$ such that Γ is abundant for all $([A'], h) \in U \times V$ with $\zeta_h(A') = 0$. This last step is performed for each $[A] \in \mathcal{M} \setminus [\theta]$, and compactness once again allows us to extract a finite subcover $U_1, \ldots, U_k$ of $\mathcal{M} \setminus [\theta]$. The proof of part (iii) is completed by setting $U = \bigcup_{i=1}^k U_i$ and $V = \bigcap_{i=1}^k V_i$.

As for part (ii), it is easiest to see this in case A is irreducible. On the other hand, if A is reducible, then similar reasoning shows that abundance is local in $\mathcal{B}^r \times \mathcal{F}_\Gamma$, but whether there exists an open neighborhood in $\mathcal{B} \times \mathcal{F}_\Gamma$ is less obvious. The following argument treats *irreducible* perturbed flat connections in a neighborhood of

A assuming A is reducible. Before continuing with the proof, we need to introduce some notation.

Since A is a fixed reducible flat connection for the rest of this proof, we write $\mathcal{K}$ for $\mathcal{K}_A$. It is useful to decompose elements $a \in \mathcal{K}$ as $a = (a_1, a_2)$ according to $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Thus $a_1 \in \Omega^1(X;\mathfrak{h})$ and $a_2 \in \Omega^1(X;\mathfrak{h}^\perp)$. For $i = 1, 2$, we have the Hodge decomposition $a_i = (a_i', a_i'')$ where $a_1' \in \mathcal{H}^1_A(X;\mathfrak{h})$ and $a_2' \in \mathcal{H}^1_A(X;\mathfrak{h}^\perp)$ are the cohomological components and a_1'', a_2'' are characterized as follows. Define $\mathcal{K}_1''$ to be the orthogonal complement of $\mathcal{H}^1_A(X;\mathfrak{h})$ in $\mathcal{K}\cap\Omega^1(X;\mathfrak{h})$, and also $\mathcal{K}_2''$ to be the orthogonal complement of $\mathcal{H}^1_A(X;\mathfrak{h}^\perp)$ in $\mathcal{K}\cap\Omega^1(X;\mathfrak{h}^\perp)$. Denote by $\Pi_i'' : \Omega^1(X;su(3)) \longrightarrow \mathcal{K}_i''$ the L^2 orthogonal projection for $i = 1, 2$. Then $a_i'' = \Pi_i'' a \in \mathcal{K}_i''$ and $a = (a_1, a_2) = (a_1', a_1'', a_2', a_2'')$. We set $\mathcal{K}'' = \mathcal{K}_1'' \oplus \mathcal{K}_2''$ and $\Pi'' = (\Pi_1'', \Pi_2'')$.

Suppose $a, b \in \Omega^1(X;su(3))$. The notation $[a\wedge b]$ indicates the product obtained by combining the wedge product on the form part with the Lie bracket on the coefficients. The following is the $su(3)$ analog of the well-known formulas for the Lie bracket in $su(2)$ (with regard to the decomposition $su(2) = u(1) \oplus u(1)^\perp$). If we decompose $a = (a_1, a_2)$ and $b = (b_1, b_2)$ according to $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as above, then

$$\begin{cases} *[a_i \wedge b_j] \in \Omega^1(X;\mathfrak{h}) & \text{if } i = j, \\ *[a_i \wedge b_j] \in \Omega^1(X;\mathfrak{h}^\perp) & \text{if } i \neq j. \end{cases}$$

The proof proceeds with two lemmas. The first one shows that the space of perturbed flat irreducible connections in X_A for small h are close to the image of the affine subspace $A + \mathcal{K}_1'' + \mathcal{H}^1_A(X;su(3))$. It also gives some control over the distance from the nearby reducibles to the affine subspace $A + \mathcal{H}^1_A(X;su(3))$ in terms of the size of the perturbation.

Lemma 3.5. *For any Γ and any $0 < R < 1$, there exist $K < \infty$ and $0 < \epsilon < 1$ such that if $A + a \in X_A$ is h-perturbed flat with $\|a\|_{L^2_1} < \epsilon$ and $\|h\|_{C^3} < \epsilon$, then*

(i) $\|a_2''\|_{L^2_1} \le R\,\|a_2'\|_{L^2_1}$ *and*

(ii) $\|a_1''\|_{L^2_1} \le R\left(\|a_1'\|_{L^2_1} + \|a_2'\|_{L^2_1}\right) + K\,\|h\|_{C^3}.$

Proof. Fix $0 < R < 1$. Consider the map from $X_A \times \mathcal{F}$ to $\mathcal{K}''$ given by $\Pi''\zeta_h(A+a)$. The linearization at $(A,0)$ restricted to $\mathcal{K}''$ with the L^2_1 norm on the domain and L^2 norm on the range is $*d_A$, an elliptic Fredholm operator with trivial kernel. Therefore there exists $\lambda > 0$ such that $\| * d_A b''\|_{L^2} \geq \lambda \|b''\|_{L^2_1}$ for all $b'' \in \mathcal{K}''$.

Now assume that $\Pi_A \zeta_h(A+a) = 0$. Expanding the equation $\Pi''_2 \zeta_h(A+a) = 0$ gives

$$0 = *d_A(a''_2) + 2\Pi_A * [a_1 \wedge a_2] - 4\pi^2 \Pi''_2 \nabla h(A+a).$$

By Taylor's theorem, the last term on the right can be replaced by

$$-4\pi^2 \left[\Pi''_2 \left(\operatorname{Hess} h(A+a_1)(a_2) + D^2 \nabla h(A + a_1 + t_1 a_2)(a_2, a_2)\right)\right],$$

for some $0 < t_1 < 1$. Here we are exploiting the equivariance of ζ_h with respect to the $\mathrm{Stab}(A)$ action. Rearranging and using the triangle inequality on $a_2 = a'_2 + a''_2$, we obtain

$$\lambda \|a''_2\|_{L^2_1} \leq \left(2C\|a_1\|_{L^2_1} + 8\pi^2 C_2 \|h\|_{C^3}\right)\left(\|a'_2\|_{L^2_1} + \|a''_2\|_{L^2_1}\right),$$

where C comes from the Sobolev multiplication theorems and C_2 is the constant given in Proposition 2.8. By shrinking ϵ to control some of the L^2_1 norms on the right side, we obtain the first claim.

To prove the second claim, expand the equation $0 = \Pi''_1 \zeta_h(A+a)$ to get

$$0 = *d_A(a''_1) + \Pi_A * ([a_1 \wedge a_1] + [a_2 \wedge a_2]) - 4\pi^2 \Pi''_1 \nabla h(A+a).$$

Rearranging, we see that

$$\lambda \|a''_1\|_{L^2_1} \leq C\left(\|a_1\|^2_{L^2_1} + \|a_2\|^2_{L^2_1}\right) + 4\pi^2 C_2 \|h\|_{C^3}.$$

Now apply the triangle inequality on the right to $a_1 = a'_1 + a''_1$ and use the first part to obtain the required bound. □

The next lemma is a similar result about tangent vectors at perturbed flat connections which are in the kernel of the Hessian of $CS + h$ (restricted to X_A). We decompose $b \in T_{A+a}X_A$ into $b = (b_1, b_2) = (b'_1, b''_1, b'_2, b''_2)$ as before.

Lemma 3.6. *For any Γ and any $0 < R < 1$, there exist $K < \infty$ and $0 < \epsilon < 1$ such that if $A + a \in X_A$ is a nonabelian h-perturbed flat with $\|a\|_{L^2_1} < \epsilon$ and $\|h\|_{C^3} < \epsilon$, and if $b \in T_{A+a}X_A$ is in the kernel of* $\mathrm{Hess}(CS + h)(A + a)$*, then*

(i) $\|b_1''\|_{L^2_1} < R\|b_1'\|_{L^2_1} + K\|a_2'\|_{L^2_1} \cdot \|b_2'\|_{L^2_1}$

(ii) $\|b_2''\|_{L^2_1} < R\|b_2'\|_{L^2_1} + K\|a_2'\|_{L^2_1} \cdot \|b_1'\|_{L^2_1}$

Proof. Setting the $\mathfrak{h}$ and $\mathfrak{h}^\perp$ components of $D\,\Pi_A\zeta_h(A + a)(b)$ equal to zero gives two coupled equations in b_1 and b_2. Expanding the $\mathfrak{h}$ component leads to

$$\begin{aligned} - * d_A b_1'' &= \Pi_A * ([a_1 \wedge b_1] + [a_2 \wedge b_2]) - 4\pi^2 \Pi_A \,\mathrm{Hess}\, h(A + a_1)(b_1) \\ &\quad -4\pi^2 \Pi_1'' D\{\mathrm{Hess}\, h(A + a_1 + t_2 a_2)(b)\}(a_2). \end{aligned}$$

Taking the L^2 norm of each side of this equation and using the various bounds as in the last lemma, it follows that

$$\lambda\|b_1''\|_{L^2_1} \leq C\left(\|a_1\|_{L^2_1} \cdot \|b_1\|_{L^2_1} + \|a_2\|_{L^2_1} \cdot \|b_2\|_{L^2_1}\right) + K\,\|h\|_{C^3}\left(\|b_1\|_{L^2_1} + \|a_2\|_{L^2_1} \cdot \|b\|_{L^2_1}\right).$$

Applying the triangle inequality, first to $b = b_1 + b_2$ and then to $b_1 = b_1' + b_1''$ everywhere on the right hand side of this equation and moving all occurrences of b_1'' to the left, we see that, for ϵ small enough,

$$\begin{aligned} \frac{\lambda}{2}\|b_1''\|_{L^2_1} &\leq 2C\left(\epsilon\,\|b_1'\|_{L^2_1} + \|a_2'\|_{L^2_1} \cdot \|b_2\|_{L^2_1}\right) + K\epsilon\left(\|b_1'\|_{L^2_1} + 2\|a_2'\|_{L^2_1} \cdot \|b_1' + b_2\|_{L^2_1}\right) \\ &\leq \epsilon\,\mathrm{const}\|b_1'\|_{L^2_1} + \mathrm{const}\|a_2'\|_{L^2_1} \cdot \|b_2\|_{L^2_1}. \end{aligned} \tag{4}$$

Similar reasoning applied to the $\mathfrak{h}^\perp$ component of $D\,\Pi_A\zeta_h(A + a)(b)$ gives

$$\frac{\lambda}{2}\,\|b_2''\|_{L^2_1} \leq \epsilon\,\mathrm{const}\|b_2'\|_{L^2_1} + \mathrm{const}\|a_2'\|_{L^2_1} \cdot \|b_1\|_{L^2_1}. \tag{5}$$

The conclusion of the lemma follows from equations (4) and (5). □

We are now ready to complete the proof of Proposition 3.4 (ii). Referring to part (i), since A is reducible, we have finite subsets $\{f_1, \ldots, f_k\}$ and $\{g_1, \ldots, g_l\}$ of $\mathcal{F}_\Gamma$ such that

(i) $\mathrm{span}\{Df_i|_{\mathcal{H}^1_A(X;\mathfrak{h})} \mid i = 1, \ldots, k\} = \mathrm{Hom}(\mathcal{H}^1_A(X;\mathfrak{h}), \mathbb{R})$,

(ii) $\mathrm{span}\{D^2 g_j|_{\mathcal{H}^1_A(X;\mathfrak{h}^\perp)^{\otimes 2}} \mid j = 1, \ldots, l\} = \mathrm{Herm}\,\mathcal{H}^1_A(X;\mathfrak{h}^\perp)$.

(iii) $Dg_j|_{\mathcal{H}^1_A(X;\mathfrak{h})} = 0$ for $j = 1, \dots, l$.

Our strategy here is to show that, given a and h sufficiently small with $A + a$ an irreducible h-perturbed flat connection, the functions $\{f_i, g_j\}$ detect all elements $b \in \ker K(A + a, h)$ to first order.

Choose a constant $N > 0$ such that, for all $u \in \mathcal{H}^1_A(X; \mathfrak{h})$ and all $v, w \in \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$, the following bounds hold:

$$\max_{1\le i\le k}\{|Df_i(A)(u)|\} \geq N\,\|u\|_{L^2_1} \tag{6}$$

$$\max_{1\le j\le l}\{|D^2 g_j(A)(v,w)|\} \geq N\,\|v\|_{L^2_1}\cdot\|w\|_{L^2_1}. \tag{7}$$

Choose ϵ small enough that these inequalities continue to hold when N is replaced by $\frac{N}{2}$ and A is replaced by $A + a$ for $\|a\|_{L^2_1} < \epsilon$.

Suppose that $h \in \mathcal{F}_\Gamma$ and that $A+a \in X_A$ is an irreducible h-perturbed flat connection, and assume $b \in \Omega^1(X; su(3))$ is an element in the kernel of Hess$(CS+h)(A+a)$. Choose functions f and g from $\{f_i\}$ and $\{g_j\}$, respectively, for which $|Df(A+a)(b_1')| \geq N/2\,\|b_1'\|_{L^2_1}$ and $|D^2 g(A+a)(a_2', b_2')| \geq N/2\,\|a_2'\|_{L^2_1}\cdot\|b_2'\|_{L^2_1}$. If either $Df(A+a)(b)$ and $Dg(A+a)(b)$ is non-zero, then we are done. So we assume both vanish and seek a contradiction.

Apply the triangle inequality to the equation $Df(A+a)(b_1') = -Df(A+a)(b_1''+b_2)$ to get the inequality

$$\begin{aligned}\frac{N}{2}\|b_1'\|_{L^2_1} &\leq |Df(A+a)(b_1'')| + |Df(A+a_1)(b_2)| + |D^2 f(A+a_1)(a_2,b_2)| \\ &+ |D^3 f(A+a_1+t_1 a_2)(a_2,a_2,b_2)|,\end{aligned}$$

where $0 < t_1 < 1$. Then $Df(A+a_1)(b_2)$ is zero by invariance under Stab$(A+a_1) \cong U(1)$, and applying bounds to the other terms gives

$$\frac{N}{2}\|b_1'\|_{L^2_1} \leq C_2\,\|f\|_{C^3}\cdot\|b_1''\|_{L^2_1} + 4C_2\,\|f\|_{C^3}\cdot\|a_2'\|_{L^2_1}\cdot\|b_2\|_{L^2_1}$$

Using Lemma 3.6, and choosing ϵ suitably small, this implies

$$\frac{N}{3}\,\|b_1'\|_{L^2_1} \leq \text{const}\,\|a_2'\|_{L^2_1}\cdot\|b_2'\|_{L^2_1}. \tag{8}$$

Next consider $Dg(A+a)(b)$. We first bound the derivative in the b_1 direction.

$$\begin{aligned}
|Dg(A+a)(b_1)| &= |Dg(A)(b_1) + D^2g(A+t_1a)(a_1,b_1) + D^2g(A+t_2a)(a_2,b_1)| \\
&= |D^2g(A+t_1a)(a_1,b_1) + D^2g(A+t_2a_1)(a_2,b_1) + D^3g(A_1)(t_2a_2,a_2,b_1)| \\
&\leq C_2\|g\|_{C^3}\cdot\|b_1\|_{L^2_1}\left(\|a_1\|_{L^2_1} + \|a_2\|^2_{L^2_1}\right) \leq \epsilon\, C_3\|b_1\|_{L^2_1} \\
&\leq \epsilon\,\text{const}\,\|b_1'\|_{L^2_1} + \epsilon\,\text{const}\,\|a_2'\|_{L^2_1}\cdot\|b_2'\|_{L^2_1}. \qquad (9)
\end{aligned}$$

In the first line, $Dg(A)(b_1)=0$ by hypothesis, and in the second, $D^2g(A+t_2a_1)(a_2,b_1)$ vanishes by gauge symmetry. The last step follows from part (i) of Lemma 3.6.

Finally, we bound the derivative of g in the b_2 direction away from zero.

$$|Dg(A+a)(b_2)| = |Dg(A+a_1)(b_2) + D^2g(A+a_1)(a_2,b_2) + D^3g(A_2)(a_2,a_2,b_2)|$$

Appling gauge symmetry once more shows that $Dg(A+a_1)(b_2)=0$ in the equation above. Bounds on the other terms give, for ϵ sufficiently small,

$$\begin{aligned}
|Dg(A+a)(b_2)| &\geq \frac{N}{2}\|a_2'\|_{L^2_1}\cdot\|b_2'\|_{L^2_1} - \text{const}\|a_2''\|_{L^2_1}\cdot\|b_2'\|_{L^2_1} \\
&\quad - \text{const}\|a_2''\|_{L^2_1}\cdot\|b_2''\|_{L^2_1} - \text{const}\|a_2'\|_{L^2_1}\cdot\|b_2''\|_{L^2_1} \\
&\geq \frac{N}{3}\|a_2'\|_{L^2_1}\cdot\|b_2'\|_{L^2_1} - \text{const}\|a_2'\|^2_{L^2_1}\cdot\|b_1'\|_{L^2_1} \qquad (10)
\end{aligned}$$

Combining inequalities (9) and (10), we get

$$\frac{N}{4}\|a_2'\|_{L^2_1}\cdot\|b_2'\|_{L^2_1} \leq \epsilon\,\text{const}\|b_1'\|_{L^2_1},$$

which, combined with inequality (8), gives the desired contradiction. □

Since X is an integral homology 3-sphere, there are no noncentral abelian flat connections. The following proposition guarantees that this, together with the property that Γ is abundant, continue to hold for small perturbations. It also provides a unique component of the flat moduli space near each perturbed flat connection, for small perturbations.

Proposition 3.7. *Suppose Γ satisfies condition (iii) of Proposition 3.4. There exists an $\epsilon_0 > 0$ such that:*

(i) *If* $A \in \mathcal{A}^* \cup \mathcal{A}^r$ *is flat and* $A' \in \mathcal{A}$ *is abelian, then* $\|A - A'\|_{L^2_1} > 2\epsilon_0$.

(ii) *If* $\|h\|_{C^3} < \epsilon_0$ *and* $A \in \mathcal{A}$ *is h-perturbed flat, then there exists* $\widehat{A} \in \mathcal{A}$ *which is flat with* $\|A - \widehat{A}\|_{L^2_1} < \epsilon_0$.

(iii) *If* $\|h\|_{C^3} < \epsilon_0$ *and* $A \in \mathcal{A}$ *is h-perturbed flat, then* Γ *is abundant for* $([A], h)$.

(iv) *If* $A, A' \in \mathcal{A}^r$ *are flat and lie on different components of the space of flat connections in* $\mathcal{A}$*, then* $\|A - A'\|_{L^2_1} > 2\epsilon_0$.

Proof. For claims (i) and (ii), see Lemma 1.3 and Proposition 1.5 of [24]. Claim (iii) follows from claim (ii). For the neighborhoods U and V in Proposition 3.4, choose ϵ_0 small enough that the ball of radius ϵ_0 around $0 \in \mathcal{F}_\Gamma$ is contained in V and the ϵ_0 neighborhood of $\mathcal{M}^* \cup \mathcal{M}^r$ in $\mathcal{B}$ is contained in U.

For part (iv), suppose to the contrary that there were no ϵ_0 satisfying the conclusion. Then we have two sequences A_i and A'_i of flat connections in $\mathcal{A}$ with $\|A_i - A'_i\|_{L^2_1} < \frac{1}{i}$ such that A_i and A'_i never lie on the same component of the space of flat connections. By compactness of $\mathcal{M}$, after passing to a subsequence, we can assume that there is a sequence of gauge transformations g_i such that $g_i \cdot A_i$ converges to a flat connection A_0. Then $g_i \cdot A'_i$ must also converge to A_0. (Note that we are using the standard gauge invariant L^2_1 norm here.)

Consequently, for i large, we see that $g_i \cdot A_i$ and $g_i \cdot A'_i$ must lie on the same component of the space of flat connections as the one containing A_0. But this implies that A_i and A'_i lie on the same component, which is a contradiction. $\square$

3.2. **Regularity theorems.** We are now ready to prove the structure theorems for $\mathcal{M}_h$ and W_ρ. We begin with the definition of regularity in this context. Throughout this subsection, Γ denotes a fixed collection of solid tori satisfying Proposition 3.4, part (iii). Thus, Γ is abundant for all pairs $([A], h) \in \mathcal{B} \times \mathcal{F}_\Gamma$ in a neighborhood of $(\mathcal{M} \setminus [\theta]) \times \{0\}$. Choose ϵ_0 as in Proposition 3.7 and define $\mathcal{F}(\epsilon_0)$ to be the ball of radius ϵ_0 about 0 in the space $\mathcal{F}_\Gamma$ of admissible functions.

Definition 3.8. *Suppose* $h \in \mathcal{F}(\epsilon_0)$ *and* $U \subset \mathcal{M}_h$ *is open. Then* U *is* **regular** *in case* $\mathcal{H}^1_{A,h}(X; su(3))$ *is trivial for all* $[A] \in U$.

Regularity as defined here makes no assumption on the irreducibility of A.

Proposition 3.9. *If $U \subset \mathcal{M}_h$ is regular, then $\mathcal{M}_h^* \cap U$ and $\mathcal{M}_h^r \cap U$ are 0-dimensional submanifolds of $\mathcal{B}^*$ and $\mathcal{B}^r$, respectively.*

Proof. This follows directly from standard Kuranishi arguments. □

We define regularity for the parameterized moduli space next. For any triple $(A, \rho, t) \in \mathcal{A} \times C^1([-1,1], \mathcal{F}(\epsilon_0)) \times [-1,1]$, define an index one Fredholm operator by the formula

$$L(A,\rho,t) \,:\, \Omega^{0+1}(X; su(3)) \oplus \mathbb{R} \longrightarrow \Omega^{0+1}(X; su(3))$$
$$(\xi, a, \tau) \mapsto K(A, \rho_t)(\xi, a) - 4\pi^2 \tau \tfrac{\partial}{\partial t} \nabla \rho_t(A)$$

Since X is an integral homology 3-sphere, the only abelian orbit in the flat moduli space is $[\theta]$, and this continues to be true for small perturbations thanks to Proposition 3.7. This explains why we dismiss the case of abelian orbits in the following definition. Note, however, that such orbits may indeed occur for large perturbations, or even for small perturbations on arbitrary 3-manifolds.

Definition 3.10. *Let $\rho : [-1,1] \longrightarrow \mathcal{F}(\epsilon_0)$ be a C^1 curve with $\mathcal{M}_{\rho(\pm 1)}$ regular. An open subset $U \subset W_\rho$ is* **regular** *if:*

(i) *$\mathcal{H}^1_{\theta,\rho_t}(X; su(3))$ is trivial for $([\theta], t) \in U$.*
(ii) *U contains no noncentral abelian orbits.*
(iii) *For all $([A], t) \in W_\rho^* \cap U$, $L(A, \rho, t)$ is surjective.*
(iv) *For all $([A], t) \in W_\rho^r \cap U$, $\Omega^{0+1}(X; \mathfrak{h}) \cap \operatorname{coker} L(A,\rho,t) = \mathcal{H}^0_A(X; su(3)) \cong u(1)$.*
(v) *There is a finite subset J of $W_\rho^r \cap U$ such that for $([A], t) \in W_\rho^r$,*

$$\dim \mathcal{H}^1_{A,\rho_t}(X; \mathfrak{h}^\perp) = \begin{cases} 2 & \text{if } ([A],t) \in J \\ 0 & \text{otherwise.} \end{cases}$$

Elements of J are called **bifurcation points***.*

(vi) *If $([A_s], t_s)$ is a parameterized curve in $W_\rho^r \cap U$ and $([A_0], t_0) \in J$, then the (multiplicity two) eigenvalue of $K(A_s, \rho(t_s))$ crosses zero transversally at $s = 0$.*

Note that regularity of W_ρ does not ensure that W_ρ is a smooth cobordism (cf. Lemma 3.11). Conditions (v) and (vi) of Definition 3.10 make sense in light of claim (i) of the next lemma.

Lemma 3.11. *If $U^r \subset W_\rho^r$ is open and $\Omega^{0+1}(X;\mathfrak{h}) \cap \operatorname{coker} L(A,\rho,t) = \mathcal{H}_A^0(X;su(3))$ for all $([A],t) \in U^r$, then U^r is a smooth 1-manifold.*

If $U \subset W_\rho$ is open and regular, then

(i) *$W_\rho^* \cap U$ and $W_\rho^r \cap U$ are both smooth 1-manifolds without boundary.*

(ii) *Each bifurcation point in U is the limit of exactly one noncompact endpoint of W_ρ^*, i.e., $J = \left(\overline{W_\rho^*} \setminus W_\rho^*\right) \cap U$.*

Proof. The first statement and (i) follow from condition (iv) of Definition 3.10 using standard Kuranishi arguments. The proof of (ii) is given below.

Fix a bifurcation point, which we assume, for simplicity of notation, to be of the form $([A],0)$. For some neighborhood $U \subset \mathcal{B} \times [-1,1]$, $W_\rho \cap U$ is the quotient by the gauge group of the zero set of the map

$$Q : X_A \times [-1,1] \longrightarrow \Omega^1(X;su(3))$$

given by $Q(A+a,t) = \Pi_A \zeta_{\rho(t)}(A+a)$.

The linearization of Q at $(A,0)$ is an elliptic Fredholm operator with index one

$$DQ_{(A,0)} : \Omega^1(X;su(3)) \oplus \mathbb{R} \longrightarrow \mathcal{K}_A$$

and $DQ_{(A,0)}(a,\tau) = \Pi_A L(A,\rho,0)(0,a,\tau)$. Fix a nontrivial $v \in \Omega^1(X;\mathfrak{h}) \oplus \mathbb{R}$ in the kernel of $DQ_{(A,0)}$. Then $\ker DQ_{(A,0)} = \operatorname{span}\{v\} \oplus \mathcal{H}^1_{A,\rho_0}(X;\mathfrak{h}^\perp)$ and $\operatorname{coker} DQ_{(A,0)} = \mathcal{H}^1_{A,\rho_0}(X;\mathfrak{h}^\perp)$.

We summarize the Kuranishi model in this situation. There is a function

$$\phi : \ker DQ_{(A,0)} \longrightarrow (\ker DQ_{(A,0)})^\perp$$

and a neighborhood $U \subset \ker DQ_{(A,0)}$ of zero such that Q restricted to the graph of $\phi|_U$ takes values in $\operatorname{coker} DQ_{(A,0)}$. Let ϕ_1 and ϕ_2 be the $\Omega^1(X;su(3))$ and $\mathbb{R}$ components of ϕ and define the map $\psi : \ker DQ_{(A,0)} \longrightarrow \mathcal{K}_A$ by setting $\psi(a,\tau) = a + \phi_1(a,\tau)$.

Now for $s \in \mathbb{R}$, define $\Psi_s : \mathcal{H}^1_{A,\rho_0}(X;\mathfrak{h}^\perp) \longrightarrow X_A$ by setting $\Psi_s(x) = A + \psi(sv + x)$. Set $CS_s(A) = CS(A) + \rho(t_s)(A)$, where $t_s = \phi_2(sv)$. Observe that $t_0 = 0$. Then for all s,

$$Q \circ \Psi_s = -4\pi^2 \nabla \left(CS_s \circ \Psi_s \right),$$

a family of gradient vector fields of $U(1)$ invariant functions on $\mathcal{H}^1_{A,\rho_0}(X;\mathfrak{h}^\perp) \cong \mathbb{C}$.

For small s, the path of orbits $([\Psi_s(0)], t_s)$ parameterizes W^r_ρ near $([A], 0)$. At the origin in $\mathcal{H}^1_{A,\rho_0}(X;\mathfrak{h}^\perp)$, the Hessian of $CS_s \circ \Psi_s$ is $\lambda_s \,\mathrm{Id}$, where λ_s is the eigenvalue referred to in condition (vi) of Definition 3.10. The proof now reduces to the parameterized Morse Lemma. See the proof of Theorem 12 in [11] for a similar argument. □

Our proof of regularity will involve considering the irreducible and reducible universal zero sets

$$Z^* = \{([A], h) \in \mathcal{B}^* \times \mathcal{F}(\epsilon_0) \mid \zeta_h(A) = 0\}$$

and

$$Z^r = \{([A], h) \in \mathcal{B}^r \times \mathcal{F}(\epsilon_0) \mid \zeta_h(A) = 0\}.$$

Within Z^r lies a subset which we hope to avoid when choosing perturbations, namely, the union over all positive integers k of

$$Z^r_k = \left\{([A], h) \in Z^r \mid \dim_{\mathbb{C}} \ker \left(K(A,h)|_{\Omega^1(X;\mathfrak{h}^\perp)} \right) = k \right\}.$$

Proposition 3.12. *The sets Z^* and Z^r are submanifolds of $\mathcal{B}^* \times \mathcal{F}(\epsilon_0)$ and $\mathcal{B}^r \times \mathcal{F}(\epsilon_0)$, respectively. For each k, Z^r_k is a submanifold of Z^r.*

Proof. Fix $([A_0], h_0) \in Z^*$. Consider the map $P : X_{A_0} \times \mathcal{F}(\epsilon_0) \longrightarrow \mathcal{K}_{A_0}$ given by $P(A,h) = \Pi_{A_0}\zeta_h(A)$. The first partial derivative $\frac{\partial P}{\partial a}(A_0,h_0)$ is Fredholm with cokernel $\mathcal{H}^1_{A_0,h_0}(X; su(3))$, but, since Γ is abundant for $([A_0], h_0)$, the image of $\frac{\partial P}{\partial h}(A_0,h_0)$ is a subspace which orthogonally projects onto this cokernel. Therefore P is a submersion at (A_0,h_0). The implicit function theorem now proves that the preimage $P^{-1}(0) \subset X_{A_0} \times \mathcal{F}(\epsilon_0)$ is smooth near (A_0,h_0), and hence Z^* is smooth near $([A_0], h_0)$.

To show smoothness of Z^r, apply the same argument to the map $P^r : X^r_{A_0} \times \mathcal{F}(\epsilon_0) \longrightarrow \mathcal{K}_{A_0} \cap \Omega^1(X;\mathfrak{h})$, which is the restriction of the map P to the reducible slice

$X^r_{A_0} = \{A_0 + a \mid a \in \mathcal{K}_{A_0} \cap \Omega^1(X;\mathfrak{h})\}$. That P^r takes values in $\mathcal{K}_{A_0} \cap \Omega^1(X;\mathfrak{h})$ follows from $\operatorname{Stab} A_0$ equivariance.

Next we treat the third case. Suppose that $([A_0], h_0) \in Z^r_k$. Define

$$\lambda_0 = \min\{|\lambda| \neq 0 \mid \lambda \in \operatorname{Spec}(K_{A_0})\}.$$

Choose a neighborhood $U \times V \subset X^r_{A_0} \times \mathcal{F}(\epsilon_0)$ of $([A_0], h_0)$ such that for $(A,h) \in U \times V$, the operator $K(A,h)$ has no eigenvalue λ with $\frac{\lambda_0}{3} < |\lambda| < \frac{2\lambda_0}{3}$. Form the small eigenspace bundle, which is the complex vector bundle E over $U \times V$ with fiber $E_{(A,h)}$ equal to

$$\operatorname{span}\left\{ u \in \Omega^{0+1}(X;\mathfrak{h}^\perp) \mid K(A,h)(u) = \lambda u \text{ where } |\lambda| < \frac{\lambda_0}{3} \right\}.$$

Let $\operatorname{Herm} E$ be the associated fiber bundle of symmetric, $\operatorname{Stab}(A_0)$ invariant (hence Hermitian) bilinear forms on E, and for each $k = 1, \ldots, \dim_{\mathbb{C}} \mathcal{H}^1_{A_0,h}(X;\mathfrak{h}^\perp)$, let $\operatorname{Herm}_k E$ be the subbundle consisting of those bilinear forms with complex rank less than or equal to $\dim_{\mathbb{C}} \mathcal{H}^1_{A_0,h}(X;\mathfrak{h}^\perp) - k$. Notice that $\operatorname{Herm}_k E$ has codimension k^2 in $\operatorname{Herm} E$.

Define $\overline{K}(A,h) : E_{(A,h)} \longrightarrow E_{(A,h)}$ to be the restriction of $K(A,h)$ to $E_{(A,h)}$ composed with the orthogonal projection to $E_{(A,h)}$, and use this to construct the section

$$R : U \times V \longrightarrow \operatorname{Herm} E \oplus \left(\mathcal{K}_{A_0} \cap \Omega^1(X;\mathfrak{h})\right)$$

given by $R(A,h) = \left(\overline{K}(A,h), P^r(A,h)\right)$. Then $Z^r_k = R^{-1}(\operatorname{Herm}_k E \oplus 0)$. Now we claim that R is a submersion at (A_0, h_0). Since the linearization of R in the first variable has cokernel $T_0 \operatorname{Herm} E_{(A_0,h_0)} \oplus \mathcal{H}^1_{A_0}(X;\mathfrak{h})$, it suffices to show that the linearization in the other variable, composed with projection to this cokernel, is onto. This is the map $T_0\mathcal{F}(\epsilon_0) \longrightarrow \operatorname{Herm} \mathcal{H}^1_{A_0}(X;\mathfrak{h}^\perp) \oplus \mathcal{H}^1_{A_0}(X;\mathfrak{h})$ given by

$$\delta h \mapsto \left(-4\pi^2 \operatorname{Hess} \delta h(A_0),\ \Pi_1' \nabla \delta h(A_0)\right),$$

where Π_1' is the projection onto $\mathcal{H}^1_{A_0}(X;\mathfrak{h})$. But surjectivity of this map follows since Γ is abundant for $([A_0], h_0)$. $\square$

We are finally ready to prove the regularity theorem for the moduli space and the parameterized moduli space. For $h_{-1}, h_1 \in \mathcal{F}(\epsilon_0)$, let $C^1([-1,1], \mathcal{F}(\epsilon_0); h_{-1}, h_1)$ denote the set of C^1 curves $\rho : [-1,1] \longrightarrow \mathcal{F}(\epsilon_0)$ with $\rho(\pm 1) = h_\pm$.

Theorem 3.13. *There exists a Baire set $\mathcal{F}(\epsilon_0)' \subset \mathcal{F}(\epsilon_0)$ such that $h \in \mathcal{F}(\epsilon_0)'$ implies $\mathcal{M}_h^* \cup \mathcal{M}_h^r$ is regular. For any $h_{-1}, h_1 \in \mathcal{F}(\epsilon_0)'$, the set of $\rho \in C^1([-1,1], \mathcal{F}(\epsilon_0); h_{-1}, h_1)$ for which W_ρ is regular is Baire.*

Proof. The projections from Z^*, Z^r, and Z_k^r to $\mathcal{F}(\epsilon_0)$ are Fredholm of index $0, 0,$ and $-k^2$, respectively. The first two index calculations simply follow from the self-adjointness of the partial derivatives in the connection variable of the maps P and P^r. The third follows easily from the second. The rest of the argument is a standard application of the Sard-Smale theorem and transversality (see [5], Section 4.3.2). $\square$

4. Orientations and Spectral Flow

In this section, we introduce orientations on the parameterized moduli space and relate them to the spectral flow of the family of operators $K(A,h)$ from the previous section. We use the index bundle of the family L to orient W_ρ^* and W_ρ^r.

The basic idea is a familiar one, used not only in 3-dimensional gauge theory by Taubes (see [24]), but also in 4-manifold gauge theory. In fact, if W_ρ were generically a cobordism, then Taubes' approach to defining an invariant would work equally well for $SU(3)$. But W_ρ is *not* generically a cobordism, as explained in Lemma 3.11, and a relationship between the orientations on W_ρ^* and W_ρ^r near a bifurcation point is provided by Theorem 4.7.

4.1. **Orientations.** Suppose that $\mathcal{F}(\epsilon_0)$ is fixed as in the previous section and consider the family of index one Fredholm operators

$$L : \mathcal{A} \times C^1([-1,1], \mathcal{F}(\epsilon_0)) \times [-1,1] \longrightarrow \mathrm{Fred}^1(\Omega^{0+1}(X; su(3)) \oplus \mathbb{R}, \Omega^{0+1}(X; su(3)))$$

introduced in subsection 3.2. The dimension of the kernel of $L(A,\rho,t)$ is not continuous in (A,ρ,t), so $\ker L$ does not form a vector bundle over $\mathcal{A} \times C^1([-1,1], \mathcal{F}(\epsilon_0)) \times [-1,1]$. Instead, we consider the index bundle of L, which is the element in the K-theory of $\mathcal{A} \times C^1([-1,1], \mathcal{F}(\epsilon_0)) \times [-1,1]$ defined by $\operatorname{ind} L = [\ker L] - [\operatorname{coker} L]$, a virtual bundle of dimension one.

Given vector spaces E and F of dimensions n and m, an orientation on $[E]-[F]$ is an orientation on the real line

$$\det([E]-[F]) = \Lambda^n E \otimes (\Lambda^m F)^*.$$

For example, if $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ are bases for E and F, then the element $(e_1 \wedge \cdots \wedge e_n) \otimes (f_1 \wedge \cdots \wedge f_m)^*$ specifies an orientation for $[E]-[F]$. More generally, if E and F are vector bundles, then an orientation on the element $[E]-[F]$ of K-theory is an orientation of the line bundle $\Lambda^n E \otimes (\Lambda^m F)^*$.

Clearly, $\operatorname{ind} L$ is orientable since the parameter space is contractable. The virtual fiber at $(\theta, 0, 0)$ is $[\mathcal{H}^0_\theta(X; su(3)) \oplus \mathbb{R}] - [\mathcal{H}^0_\theta(X; su(3))]$, and our convention for orienting ind L is to propagate the canonical orientation at $(\theta, 0, 0)$ given by

$$(v_1 \wedge \cdots \wedge v_8 \wedge w) \otimes (v_1 \wedge \cdots \wedge v_8)^*, \tag{11}$$

where $\{v_1, \ldots, v_8\}$ is a basis for $su(3) = \mathcal{H}^0_\theta(X; su(3))$ and w is a tangent vector to $[-1, 1]$ at $t = 0$ pointing in the positive direction.

Suppose that $\rho \in C^1([-1, 1], \mathcal{F}(\epsilon_0))$ and W_ρ is regular. Then W^*_ρ inherits an orientation because of the natural identification $T_{([A],t)} W^*_\rho \cong \ker L(A, \rho, t)$. There is also an induced orientation for W^r_ρ, but this is less obvious. First, suppose $([A], t) \in W^r_\rho$ is not a bifurcation point. An orientation is given by declaring that a nontrivial vector $v \in T_{([A],t)} W^r_\rho$ is positively oriented if the element $(u \wedge v) \otimes u^* \in \det \operatorname{ind} L(A, \rho, t)$ agrees with the orientation of ind L for any $u \in u(1) \cong \mathcal{H}^0_A(X; su(3))$.

Now suppose that $([A], t) \in W^r_\rho$ is a bifurcation point. The dimension of $\ker L$ and $\operatorname{coker} L$ both jump by two at (A, ρ, t), but we obtain an orientation consistent with the one above by requiring that $(u \wedge x \wedge y \wedge v) \otimes (u \wedge x \wedge y)^*$ agree with the given orientation on ind $L(A, \rho, t)$, where $\{x, y\}$ is a basis for $\mathcal{H}^1_{A,h}(X; \mathfrak{h}^\perp)$, the new part of the kernel (and cokernel) of L at (A, ρ, t).

4.2. **Spectral flow.** In analogy with Taubes' gauge theoretic description of the Casson invariant, our formula will involve counting irreducible perturbed flat orbits with sign according to their spectral flow. We adopt the following convention for computing the spectral flow.

Definition 4.1. *Suppose $\mathcal{U}$ is a real, infinite dimensional, separable Hilbert space and $K : [0, 1] \longrightarrow \mathrm{SAFred}(\mathcal{U})$ is a continuously differentiable family of self-adjoint Fredholm operators with discrete spectrum on $\mathcal{U}$. Note that the eigenvalues of K_t vary continuously differentiably. Choose δ such that*

$$0 < \delta < \inf\{|\lambda| \neq 0 \mid \lambda \in \operatorname{Spec} K_0 \cup \operatorname{Spec} K_1\}.$$

The **spectral flow** *along K_t from K_0 to K_1, denoted $Sf(K_0, K_1)$, is the intersection number, in $[0, 1] \times \mathbb{R}$, of the graphs of the eigenvalues of K_t, counted with multiplicities,* *with the line segment from* $(0,-\delta)$ *to* $(1,\delta)$*. It is a homotopy invariant of the path* K_t *relative to its endpoints.*

Note that with this convention for counting zero modes,

$$Sf(K_0, K_1) + Sf(K_1, K_2) = Sf(K_0, K_2) - \dim\ker K_1.$$

We are primarily interested in the spectral flow of the operator $K(A,h)$ from subsection 2.2. Completing $\Omega^{0+1}(X; su(3))$ in the L^2 norm, we regard $K(A,h)$ as a family of self-adjoint Fredholm operators on $\Omega^{0+1}(X; su(3))$ with dense domain the space of L^2_1 forms,

$$K : \mathcal{A} \times \mathcal{F}(\epsilon_0) \longrightarrow \text{SAFred}\left(\Omega^{0+1}(X; su(3))\right).$$

Define $\deg : \mathcal{G} \longrightarrow \mathbb{Z}$ by setting $\deg g = \deg g'$, where $g' : X \longrightarrow SU(2)$ is a map homotopic to g. That $\deg g$ is well-defined follows from the next proposition, which can be proved by noting that $SU(n)$ is homotopy equivalent to a CW-complex with 3-skeleton S^3 and the next lowest cell in dimension 5.

Proposition 4.2. *Fix* $n > 2$ *and consider the standard inclusion* $i : SU(2) \subset SU(n)$.

(i) *If* $g \in C^\infty(X, SU(n))$*, then there exists* $g' : X \longrightarrow SU(2)$ *with* $i \circ g' \simeq g$.

(ii) *If* $g_0, g_1 \in C^\infty(X, SU(2))$ *with* $i \circ g_0 \simeq i \circ g_1$*, then* $g_0 \simeq g_1$.

Proposition 4.2 gives the following formula for the spectral flow between two gauge equivalent connections.

Proposition 4.3. (i) *The spectral flow of* $K(A,h)$ *along a path* (A_t, h_t) *is independent of the path connecting* (A_0, h_0) *to* (A_1, h_1).

(ii) *The spectral flow of* K *from* (A,h) *to* (gA, h) *equals* $12\deg g - \dim\ker K(A,h)$.

Proof. Part (i) follows since $\mathcal{A} \times \mathcal{F}(\epsilon_0)$ is contractable. Part (ii) follows by an index computation, the point being that spectral flow around a closed path in $\mathcal{A}$ equals the index of the self-duality operator on $SU(3)$ connections over $X \times S^1$. Details can be found in [13]. □

Remark. Suppose $A \in \mathcal{A}^r$. By applying a gauge transformation, we can assume that $A \in \mathcal{A}_{S(U(2)\times U(1))}$. Consider now the standard decomposition of the Lie algebra $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ given by the action of $\operatorname{Stab} A = U(1)$ and split the operator $K(A,h)$ accordingly. Because θ and A can be connected by a path in $\mathcal{A}_{S(U(2)\times U(1))}$, the spectral flow of K from $(\theta, 0)$ to (A,h) splits as

$$Sf(\theta, A) = Sf_{\mathfrak{h}}(\theta, A) + Sf_{\mathfrak{h}^\perp}(\theta, A).$$

Notice that $U(1)$ equivariance of $K(A,h)$ implies that $Sf_{\mathfrak{h}^\perp}(\theta, A)$ is divisible by two. Using part (ii) of the previous proposition and the well-known, analogous result (for $su(2)$) that $Sf_{\mathfrak{h}}(A, gA) = 8 \deg g - \dim\ker K_A|_{\Omega^{0+1}(X;\mathfrak{h})}$, we see that

$$Sf_{\mathfrak{h}^\perp}(A, gA) = 4 \deg g - \dim\ker K_A|_{\Omega^{0+1}(X;\mathfrak{h}^\perp)}.$$

4.3. **The relationship between orientations and spectral flow.** There is a fundamental relationship between the orientation of the one dimensional virtual bundle $\operatorname{ind} L$ and the spectral flow of $K(A,h)$. We describe it next, in some generality.

Suppose that $\mathcal{U}$ is an infinite dimensional, separable Hilbert space and that Z is a connected, simply connected parameter space. Let

$$K : Z \longrightarrow \operatorname{SAFred}(\mathcal{U})$$

be a parameterized family of self-adjoint Fredholm operators on $\mathcal{U}$ and $v : Z \longrightarrow \mathcal{U}$ be a continuous map.

Define $L_z : \mathcal{U} \oplus \mathbb{R} \longrightarrow \mathcal{U}$ by $L_z(u, \tau) = K_z(u) + \tau v_z$ for $(u, \tau) \in \mathcal{U} \oplus \mathbb{R}$. Clearly $L_z \in \operatorname{Fred}^1(\mathcal{U} \oplus \mathbb{R}, \mathcal{U})$. For any $z \in Z$, let $\Pi_z : \ker L_z \longrightarrow \mathbb{R}$ be the projection onto $\mathbb{R}$ and $\Pi_{\ker L_z} : \mathcal{U} \oplus \mathbb{R} \longrightarrow \ker L_z$ be the projection onto the $\ker L_z$.

Suppose that $z_0 \in Z$ is a fixed base point and $v_{z_0} = 0$. Choose an orientation $\mathcal{O}$ for $\operatorname{ind} L$ by the convention in equation (11), and let $\mathcal{O}_z$ denote the induced orientation on $\operatorname{ind} L_z$. If L_z is surjective, then $\mathcal{O}_z$ gives an orientation of $\ker L_z$. Notice that whenever K_z is an isomorphism, $\ker L_z$ is spanned by $(-K_z^{-1}(v_z), 1)$. In this case, and more generally when $v_z \perp \ker K_z$, the spectral flow of K_z allows us to compare $\mathcal{O}_z$ with another natural orientation on $\ker L_z$.

Proposition 4.4. *Suppose* $v_{z_1} \perp \ker K_{z_1}$. *Then if* $\{u_1, \ldots, u_k\}$ *is a basis for* $\ker K_{z_1}$, $\{u_1, \ldots, u_k, \left(-K_{z_1}^{-1}(v_{z_1}), 1\right)\}$ *is a basis for* $\ker L_{z_1}$. *Furthermore, the orientation on* $\operatorname{ind} L_{z_1}$ *agrees with*

$$(-1)^{Sf(K_{z_0}, K_{z_1})} \left(u_1 \wedge \ldots \wedge u_k \wedge \left(-K_{z_1}^{-1}(v_{z_1}), 1\right)\right) \otimes (u_1^* \wedge \ldots \wedge u_k)^*.$$

Remark. K_{z_1} is an isomorphism if and only if Π_z is an isomorphism, and then the proposition states that the orientation on $\ker L_{z_1}$ is $(-1)^{Sf(K_{z_0}, K_{z_1})} \Pi_{z_1}^* \mathcal{O}_{\mathbb{R}}$.

Proof. The first claim is obvious. The proof of the second goes as follows. Connect z_0 to z_1 by a path z_t. By [12], K_{z_t} is homotopic relative its endpoints to a path K_t in $\mathrm{SAFred}(\mathcal{U})$ so that there is a finite set $\{t_1, \ldots, t_k\} \subset (0, 1)$ such that

$$\dim \ker K_t = \begin{cases} 1 & \text{if } t \in \{t_1, \ldots, t_k\} \\ 0 & \text{otherwise.} \end{cases}$$

We can further assume that any eigenvalue of K_t which crosses zero does so transversely.

Similarly, v_{z_t} can be homotoped relative to its endpoints to a path v_t in $\mathcal{U}$ such that the path L_t in $\mathrm{Fred}^1(\mathcal{U} \oplus \mathbb{R}, \mathcal{U})$ defined by $L_t(u, \tau) = K_t(u) + \tau v_t$ is surjective for all $t \in (0, 1]$. Let $\mathcal{O}_t$ be the orientation on $\ker L_t$ coming from $\mathcal{O}_0 = \mathcal{O}_{z_0}$.

Fix a t_j with $\ker K_{t_j}$ nontrivial. For $t \in (t_j - \delta, t_j + \delta)$, let λ_t be the eigenvalue of K_t which crosses zero when $t = t_j$. Choose u_t to be a unit eigenvector with eigenvalue λ_t so that $K_t(u_t) = \lambda_t \cdot u_t$.

For $t \in (t_j - \delta, t_j + \delta)$, we have an orthogonal decomposition of $\mathcal{U}$ into $\mathcal{U}_t' \oplus \mathcal{U}_t''$ where $\mathcal{U}_t'' = \mathrm{span}\{u_t\}$ and $\mathcal{U}_t'$ is its orthogonal complement. Set $a_t = \langle u_t, v_t \rangle$, $v_t' = v_t - a_t u_t$ and $K_t'(w) = K(w) - \lambda_t \langle u_t, w \rangle u_t$ for $w \in \mathcal{U}$. Note that K_t' is invertible on $\mathcal{U}_t'$ and set $w_t = a_t u_t + \lambda_t (K_t')^{-1} v_t'$. The vector $(w_t, -\lambda_t)$ spans $\ker L_t$ for $t \in (t_j - \epsilon, t_j + \epsilon)$. Since the inner product $\langle (w_t, -\lambda_t), (0, 1) \rangle$ changes sign at t_j, it follows that the orientation of $\Pi_t^* \mathcal{O}_{\mathbb{R}}$ changes relative to $\mathcal{O}_t$ at t_j. Such a change occurs for each t_j, which is where $Sf(K_0, K_t)$ changes by ± 1.

This proves the second claim in case $\ker K_{z_1}$ is trivial. For the general case, we may assume that all the eigenvalues of K_t which approach zero as $t \to 1^-$ are negative for t near 1. This implies that $Sf(K_0, K_1) = Sf(K_0, K_t)$ for $t \in (1-\delta, 1)$. We then claim that the orientation given by $\Pi_{\ker L_t}\big(-K_{z_t}^{-1}(v_{z_t}), 1\big)$ propagates to $\big(u_1 \wedge \ldots \wedge u_k \wedge \big(-K_{z_1}^{-1}(v_{z_1}), 1\big)\big) \otimes (u_1 \wedge \ldots \wedge u_k)^*$.

Recall our convention for propagating the orientation of $\operatorname{ind} L$ across a point where the $\dim \ker L$ jumps. The orientation given by $\Pi_{\ker L_t}\big(-K_{z_1}^{-1}(v_{z_1}), 1\big)$ propagates to

$$\big(\big(-K_{z_1}^{-1}(v_{z_1}), 1\big) \wedge u_1 \wedge \ldots \wedge u_k\big) \otimes \Big(\widetilde{L}_t(u_1) \wedge \ldots \wedge \widetilde{L}_t(u_k)\Big)^*,$$

where $\widetilde{L}_t = \Pi_{\operatorname{coker} L_1} \circ L_t$. Since $\widetilde{L}_t$ is negative definite on $\operatorname{span}\{u_1, \ldots, u_k\}$, it follows that $\widetilde{L}_t(u_1) \wedge \ldots \wedge \widetilde{L}_t(u_k)$ is proportional to $(-1)^k u_1 \wedge \ldots \wedge u_k$. Permuting the $\big(-K_{z_1}^{-1}(v_{z_1}), 1\big)$ factor past all the u_i's introduces another $(-1)^k$ which cancels with the first. □

Applying Proposition 4.4 to the oriented strata in a regular moduli space gives the following corollary.

Corollary 4.5. *Assume that $\rho : [-1, 1] \longrightarrow \mathcal{F}(\epsilon_0)$ is a path of perturbations such that $\mathcal{M}_{\rho(+1)}$, $\mathcal{M}_{\rho(-1)}$, and W_ρ are all regular. Then -1 and $+1$ are regular values of the projections from W_ρ^* and W_ρ^r to $[-1, 1]$. Suppose $\varepsilon = \pm 1$ and $([A], \varepsilon) \in \mathcal{M}_{\rho(\varepsilon)}^* \cup \mathcal{M}_{\rho(\varepsilon)}^r$, and set $s = Sf(K_{\theta,0}, K_{A,\rho(\varepsilon)})$. Then the boundary orientation of W_ρ^* or W_ρ^r at $([A], \varepsilon)$ equals $(-1)^s$ if $\varepsilon = 1$ and it equals $-(-1)^s$ if $\varepsilon = -1$.*

Proof. Note that the boundary orientation at $([A], \varepsilon)$ is positive if and only if the orientation on the 1-dimensional stratum of W_ρ at $([A], \varepsilon)$ agrees with $\varepsilon \Pi^* \mathcal{O}_{\mathbb{R}}$. In the irreducible case, $K(A, \rho(\varepsilon))$ is an isomorphism, so the remark following Proposition 4.4 proves the claim.

The reducible case also follows by a direct application of Proposition 4.4, letting $-4\pi^2 \frac{\partial}{\partial t} \nabla \rho_t(A)\big|_{t=\varepsilon}$ play the role of the v_{z_1} for the operator $L(A, \rho, t)$ and observing that this vector is orthogonal to $\ker K(A, \rho(\varepsilon)) = \mathcal{H}_A^0(X; su(3))$. □

4.4. **Orientations near a bifurcation point.** In this subsection, we identify the boundary orientation of a bifurcation point with the oriented $\mathfrak{h}^\perp$ spectral flow of $K(A,h)$ along W^r_ρ across this point. The precise relationship is given in Lemma 4.6. This is the crucial observation needed for Theorem 4.7, which is used in section 5 to show that our invariant is well-defined.

Consider the operator $L(A,\rho,t) : \Omega^{0+1}(X;su(3)) \oplus \mathbb{R} \longrightarrow \Omega^{0+1}(X;su(3))$ for a fixed $\rho \in C^1([-1,1],\mathcal{F}(\epsilon_0))$ such that W_ρ, $\mathcal{M}_{\rho(-1)}$, and $\mathcal{M}_{\rho(+1)}$ are regular. Suppose that W_ρ has a bifurcation point, which we take to be $([A],0)$ for simplicity of notation. Assume that $A \in \mathcal{A}_{S(U(2)\times U(1))}$ is a representative of the orbit $[A]$. Choose a covariantly constant, diagonal $su(3)$-valued 0-form

$$u = \begin{pmatrix} i/3 & & 0 \\ & i/3 & \\ 0 & & -2i/3 \end{pmatrix} \in \mathcal{H}^0_A(X;su(3)).$$

Then the complex structure J on $\Omega^{0+1}(X;\mathfrak{h}^\perp)$ is given by $\exp(\pi u/2) \in \operatorname{Stab} A$ acting by conjugation, i.e., $Jx = [u,x]$ for $x \in \Omega^{0+1}(X;\mathfrak{h}^\perp)$. Choose a nonzero $x \in \mathcal{H}^1_{A,\rho_0}(X;\mathfrak{h}^\perp)$ and set $y = Jx$.

Let $v \in \Omega^1(X;\mathfrak{h}) \oplus \mathbb{R}$, be an element of $\ker L(A,\rho,0)$ such that

$$(u \wedge x \wedge y \wedge v) \otimes (u \wedge x \wedge y)^* \tag{12}$$

is the orientation for $\operatorname{ind} L$ at $(A,\rho,0)$. In other words, v is an oriented tangent vector for W^r_ρ.

Solutions to the equation $\zeta_{\rho(t)}(A') = 0$ near $(A,0)$ in $X_A \times [-1,1]$ take the form $(A',t) = (A + sx + o(s^2), o(s^2))$, $s > 0$, up to the action of $\operatorname{Stab} A$. For such a nearby solution, x projects nontrivially into the 1-dimensional kernel of $L(A',\rho,t)$ (this follows from Lemma 3.11) and its image, thought of as a tangent vector to W^*_ρ, points away from the endpoint.

We shall now compare the orientation of $\operatorname{ind} L$ at (A',ρ,t) with that given by x. To do so, we consider

$$L'(\xi,a,\tau) = \tfrac{\partial}{\partial s} L(A+sx,\rho,0)(\xi,a,\tau)\big|_{s=0},$$

where the map on the right is restricted to $\ker L(A, \rho, 0)$ and then projected onto $\operatorname{coker} L(A, \rho, 0)$.

One can check that $\dim \ker L' = 1$, and so the orientation on $\ker L(A', \rho, t)$ points in the direction of ϵx, where $\epsilon = \pm 1$ is such that

$$(\epsilon x \wedge u \wedge v \wedge y) \otimes (L'(u) \wedge L'(v) \wedge L'(y))^* \tag{13}$$

is the orientation for $\operatorname{ind} L$ at $(A, \rho, 0)$. The following lemma is the key step in proving Theorem 4.7 because it identifies this ϵ in terms of the $\mathfrak{h}^\perp$ spectral flow of $K(A, h)$.

Lemma 4.6. *Suppose that x, y, u are chosen as above. Denote by L_1' the composition of L' with the projection onto $\Omega^1(X; su(3))$. Then*

(i) $L'(u) = -y$ *and* $L'(y) = -u$, *and*

(ii) $L_1'(v) = D_v(*d_{A',\rho_t}(x))|_{(A',t)=(A,0)}$.

Remark. Recall from section 2 that $*d_{A,h} = *d_A - 4\pi^2 \operatorname{Hess} h(A)$. The notation D_v means the derivative as (A, t) is varied with tangent vector v.

Proof. First we compute that

$$\begin{aligned} L'(u) &= \tfrac{\partial}{\partial s} L(A + sx, 0)(u, 0, 0)\big|_{s=0} \\ &= \tfrac{\partial}{\partial s} d_{A+sx}(u)\big|_{s=0} \\ &= [x, u] = -[u, x] = -Jx = -y. \end{aligned}$$

A similar computation yields $L'(y) = -u$, and these together prove (i).

Claim (ii) follows by commuting mixed partials as follows. Let (a, τ) denote the components of v in $\Omega^1(X; \mathfrak{h}) \oplus \mathbb{R}$.

$$\begin{aligned} L_1'(a, \tau) &= \tfrac{\partial}{\partial s} \left(*d_{A+sx}(a) - 4\pi^2 \operatorname{Hess} \rho_0(A + sx)(a) - 4\pi^2 \tau \tfrac{\partial}{\partial t} \nabla \rho_t(A + sx)\big|_{t=0} \right)\big|_{s=0} \\ &= \tfrac{\partial}{\partial s} \tfrac{\partial}{\partial r} \left(*F(A + sx + ra) - 4\pi^2 \nabla \rho_{r\tau}(A + sx + ra) \right)\big|_{(r,s)=(0,0)} \\ &= D_{(a,\tau)} \left(\tfrac{\partial}{\partial s} \left(*F(A' + sx) - 4\pi^2 \nabla \rho_t(A' + sx) \right)\big|_{s=0} \right)\big|_{(A',t)=(A,0)} \\ &= D_{(a,\tau)}(*d_{A',\rho_t}(x))\big|_{(A',t)=(A,0)} \end{aligned}$$

This completes the proof of (ii). ∎

Using part (i) of Lemma 4.6 and comparing the two orientations for ind L at $([A], 0)$ given in equations (12) and (13), we see that ϵ has the opposite sign of the inner product $\langle L'(v), x\rangle$, where v is the oriented vector tangent to W^r_ρ at $([A], 0)$. From part (ii) of the lemma, it follows that $\langle L'(v), x\rangle$ has the same sign as the derivative of the path of (multiplicity two) eigenvalues of $K(A + ra, \rho_{r\tau})$ which crosses zero at $r = 0$. Figure 2 illustrates what this means in terms of spectral flow. Here W^r_ρ is the dotted line and W^*_ρ the solid one. In the diagram on the left, $Sf_{\mathfrak{h}^\perp}(A_-, A_+) = -2$ and in the one on the right, $Sf_{\mathfrak{h}^\perp}(A_-, A_+) = 2$.

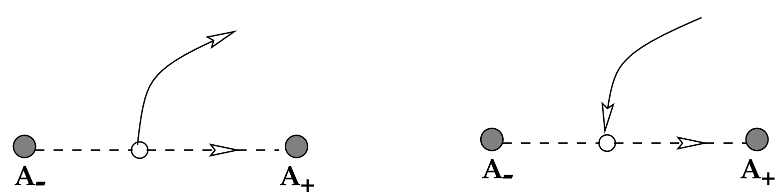

FIGURE 2. A neighborhood of a bifurcation point in W_ρ.

Caution. The operator $K(A, h)$ on $\Omega^{0+1}(X; \mathfrak{h}^\perp)$ is equivariant with respect to the action of $\operatorname{Stab} A \cong U(1)$, thus it is Hermitian with respect to J. Viewed this way, the eigenvalue here would have (complex) multiplicity one, but in order to avoid confusion, we regard $K(A, h)$ on (0+1)-forms with values in either component of the splitting $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as a *real* operator.

Summing over all the bifurcation points results in the following theorem.

Theorem 4.7. *Let $\rho \in C^1([-1, 1], \mathcal{F}(\epsilon_0))$ be a curve with $W_\rho, \mathcal{M}_{\rho(\pm 1)}$ regular and suppose C is a connected component of W^r_ρ. Define $b(C)$ to be the number of bifurcation points on C counted with orientation as boundary points of $\overline{W^*_\rho}$.*

(i) *If $\partial C = \emptyset$, then $b(C) = 0$.*

(ii) *If $\partial C = ([A_+], \varepsilon_+) \cup -([A_-], \varepsilon_-)$, where $\varepsilon_\pm \in \{-1, +1\}$ are not necessarily distinct, then $b(C) = \frac{1}{2} Sf_{\mathfrak{h}^\perp}(A_-, A_+)$, provided the representatives $A_\pm$ for $[A_\pm]$ are chosen to lie on the same component of a lift of C to $\mathcal{A}^r$.*

Proof. To prove (i), suppose C is a component of W^r_ρ with $\partial C = \emptyset$. Choose a path of connections A_s and perturbations h_s for $s \in [0, 1]$ such that $([A_s], h_s)$ parameterizes

C. Then $A_1 = gA_0$ for some $g \in \mathcal{G}$. Proposition 3.7 implies that the entire path A_s of perturbed flat connections lies within ϵ_0 of some component K of the space of flat connections upstairs in $\mathcal{A}$. By our choice of ϵ_0, that proposition also shows that A_s does not come within ϵ_0 of any other component of the flat connections. But if A_0 is within ϵ_0 of K, then $gA_0 = A_1$ is within ϵ_0 of gK since we are using the standard gauge invariant L^2_1 metric. Thus $gK = K$.

Now $CS : \mathcal{A} \longrightarrow \mathbb{R}$ is constant on connected components of the space of flat connections, and this implies that $g \in \mathcal{G}_0$, the connected component of the identity in $\mathcal{G}$, since otherwise $\deg g \neq 0$. Therefore, using the relationship between spectral flow and degree described in the remark following Proposition 4.3, we get

$$b(C) = \tfrac{1}{2}\, Sf_{\mathfrak{h}^\perp}(K_{A_0,h_0}, K_{A_1,h_1}) = 2(CS(A_1) - CS(A_0)) = 0.$$

This proves (i), and part (ii) of the theorem is clear. $\square$

Example. We indicate briefly the consequence of the above theorem for the situation illustrated in Figure 1. First of all, part (i) of Theorem 4.7 implies that the $\mathfrak{h}^\perp$ spectral flow around the one closed component equals 0.

Along the other components, which are the two other dotted curves, the $\mathfrak{h}^\perp$ spectral flow in the oriented direction equals 2 for the component on top and it equals 4 for the one on bottom. In other words, the $\mathfrak{h}^\perp$ spectral flow along the bottom component from *left* to *right* equals -4.

5. The Invariant

In this section, we define the invariant $\lambda_{SU(3)}(X)$ for X an orientable, integral homology 3-sphere. Choose an orientation and Riemannian metric on X, as well as a collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of embedded solid tori in X satisfying the conlusion of Proposition 3.4. Let $\mathcal{F}(\epsilon_0)$ be the ϵ_0 neighborhood of 0 in $\mathcal{F}_\Gamma$, where ϵ_0 is given by Proposition 3.7. Then choose a perturbation $h \in \mathcal{F}(\epsilon_0)$ so that $\mathcal{M}_h$ is regular. By Proposition 3.9, $\mathcal{M}_h^*$ is a compact 0-manifold. We would like to define an invariant of X by counting the points $[A] \in \mathcal{M}_h^*$ with sign according to the parity of the spectral flow of K. This integer, however, depends on the choice of perturbation $h \in \mathcal{F}(\epsilon_0)$ and in order to obtain a well-defined invariant, we must include a correction term determined from $\mathcal{M}_h^r$.

When the perturbation h is clear from the context, we let $Sf(A_0, A_1)$ be an abbreviation for $Sf(K_{A_0,h}, K_{A_1,h})$. For $A_0, A_1 \in \mathcal{A}^r$, the spectral flow splits as

$$Sf(A_0, A_1) = Sf_{\mathfrak{h}}(A_0, A_1) + Sf_{\mathfrak{h}^\perp}(A_0, A_1)$$

according to the decomposition $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$.

Theorem 5.1. *Suppose that $h \in \mathcal{F}(\epsilon_0)$ with $\mathcal{M}_h$ is regular. Pick gauge representatives A for each orbit $[A] \in \mathcal{M}_h$, and for each representative of a reducible orbit $[A] \in \mathcal{M}_h^r$, choose also a flat connection $\widehat{A}$ with $\|A - \widehat{A}\|_{L_1^2} < \epsilon_0$. The quantity*

$$\sum_{[A]\in\mathcal{M}_h^*} (-1)^{Sf(\theta,A)} - \frac{1}{2} \sum_{[A]\in\mathcal{M}_h^r} (-1)^{Sf(\theta,A)} (Sf_{\mathfrak{h}^\perp}(\theta, A) - 4\, CS(\widehat{A})).$$

is independent of choice of representatives A for $[A]$ in both sums and independent of the choice of h.

Proof. Note that the existence of $\widehat{A}$ is guaranteed by Proposition 3.7. We first argue that the quantity is independent of the representatives A chosen for the orbits $[A] \in \mathcal{M}_h^r$. Write $\lambda'(h) = \sum_{[A]\in\mathcal{M}_h^*} (-1)^{Sf(\theta,A)}$ for the first sum and $\lambda''(h) = \frac{1}{2}\sum_{[A]\in\mathcal{M}_h^r} (-1)^{Sf(\theta,A)} (Sf_{\mathbb{C}^2}(\theta, A) - 4\, CS(\widehat{A}))$ for the second. Part (ii) of Proposition 4.3 shows that $\lambda'(h)$ is independent of choice of the representatives A for $[A] \in \mathcal{M}_h^*$.

Also, for a fixed representative A for $[A] \in \mathcal{M}_h^r$, if $\widehat{A}$ and $\widehat{A}'$ are both flat connections in an ϵ_0 neighborhood of A, then part (iv) of Proposition 3.7 implies that $\widehat{A}$ and $\widehat{A}'$ lie on the same component of the flat connections, hence $CS(\widehat{A}) = CS(\widehat{A}')$. Now suppose $g \in \mathcal{G}$. Then, by the remark following Proposition 4.3, $Sf_{\mathfrak{h}^\perp}(A, gA) = 4 \deg g$. Since $CS(g\widehat{A}) = CS(\widehat{A}) + \deg g$, $\lambda''(h)$ is also independent of the choice of representatives A for $[A] \in \mathcal{M}_h^r$.

We now argue that the above quantity is independent of choice of h. Suppose that h_- and h_+ are admissible functions in $\mathcal{F}(\epsilon_0)$ and that $\mathcal{M}_{h_\pm}$ are both regular. Set $\mathcal{M}_\pm = \mathcal{M}_{h_\pm}$ and connect h_- and h_+ by a path $\rho : [-1,1] \to \mathcal{F}(\epsilon_0)$ with $\rho(\pm 1) = h_\pm$ so that the parameterized moduli space W_ρ is regular.

Compactify the irreducible stratum W_ρ^* by adding bifurcation points and denote the compact, oriented 1-manifold with boundary so obtained by $\overline{W_\rho^*}$. Of course, the total number of boundary points, counted with boundary orientation, equals zero. Every boundary point which is not a bifurcation point can be identified with a point in the disjoint union $\mathcal{M}_-^* \cup \mathcal{M}_+^*$. The orientations of these points are described by Corollary 4.5, as follows. For $[A] \in \mathcal{M}_+^*$, the boundary orientation of W_ρ^* at $([A], +1)$ is $(-1)^{Sf(\theta,A)}$, while for $[A] \in \mathcal{M}_-^*$, the boundary orientation of $([A], -1)$ at W_ρ^* is $-(-1)^{Sf(\theta,A)}$. Therefore $\lambda'(h_+) - \lambda'(h_-)$ equals minus the number of bifurcation points counted with orientation as boundary points of $\overline{W_\rho^*}$.

It remains to show that this algebraic sum of bifurcation points equals $\lambda''(h_+) - \lambda''(h_-)$. To prove this, we invoke Theorem 4.7. By part (i), the closed components of W_ρ^r do not contribute to this sum, so suppose that C is a component of W_ρ^r and $\partial C = ([A_+], \varepsilon_+) \cup -([A_-], \varepsilon_-)$, where $\varepsilon_\pm \in \{-1, 1\}$. Let $b(C)$ be the algebraic sum of bifurcation points on C. Since $\lambda''(h_+)$ and $\lambda''(h_-)$ are both independent of the choice of representatives A for $[A]$, we can choose A_+ and A_- to lie on the same component of the lift of C to $\mathcal{A}^r$. Thus $CS(\widehat{A_+}) = CS(\widehat{A_-})$. By part (ii) of Theorem 4.7,

$$b(C) = \tfrac{1}{2}\, Sf_{\mathfrak{h}^\perp}(A_-, A_+) = \tfrac{1}{2} \left(Sf_{\mathfrak{h}^\perp}(\theta, A_+) - Sf_{\mathfrak{h}^\perp}(\theta, A_-) \right).$$

On the other hand, the contribution to $\lambda''(h_+) - \lambda''(h_-)$ from the endpoints of C is

$$\tfrac{1}{2}\left[\varepsilon_+(-1)^{Sf(\theta,A_+)}\, Sf_{\mathfrak{h}^\perp}(\theta, A_+) + \varepsilon_-(-1)^{Sf(\theta,A_-)}\, Sf_{\mathfrak{h}^\perp}(\theta, A_-)\right].$$

It is important to keep in mind that $\varepsilon_\pm$ need not be distinct; several possibilities are pictured in Figure 1. Now the reducible case of Corollary 4.5 implies that $\varepsilon_+ = (-1)^{Sf(\theta,A_+)}$ and $\varepsilon_- = (-1)^{Sf(\theta,A_-)}$, and this completes the proof. □

The quantity in Proposition 5.1 is seen to be independent of the choice of metric on X by the same argument as was used for Proposition 2.3 of [24]. That it is also independent of the choice of Γ is an exercise which we leave for the reader.

Definition 5.2. *Suppose that $h \in \mathcal{F}(\epsilon_0)$ and that $\mathcal{M}_h$ is regular. Define the* **SU(3) Casson invariant** *by*

$$\lambda_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}_h^*} (-1)^{Sf(\theta,A)} - \frac{1}{2}\sum_{[A]\in\mathcal{M}_h^r} (-1)^{Sf(\theta,A)}(Sf_{\mathfrak{h}^\perp}(\theta, A) - 4\, CS(\widehat{A}) + 2).$$

By Theorem 5.1, this gives a well-defined invariant of integral homology 3-spheres.

Notice that the last term in the second sum above simply adds a multiple of the $SU(2)$ Casson invariant. This part of $\lambda_{SU(3)}(X)$ is independent of h by the argument given in [24]. Therefore, the previous theorem implies that $\lambda_{SU(3)}(X)$ is independent of $h \in \mathcal{F}(\epsilon_0)$. The following proposition explains why we have chosen to normalize $\lambda_{SU(3)}(X)$ this way.

Proposition 5.3. (i) *If $\pi_1 X = 0$, then $\lambda_{SU(3)}(X) = 0$.*
(ii) $\lambda_{SU(3)}(-X) = \lambda_{SU(3)}(X)$.

Proof. Part (i) is obvious. To prove (ii), observe that

$$Sf_{-X}(A_0, A_1) = -\, Sf_X(A_0, A_1) - (\dim\ker K_{A_0} + \dim\ker K_{A_1}),$$

where the subscript indicates a choice of orientation on X. This is equally valid for $\mathfrak{h}^\perp$ coefficients in case A_0 and A_1 are reducible. Applying this to all three spectral flows appearing in the definition of $\lambda_{SU(3)}(-X)$ and noting further that $CS_{-X}(\widehat{A}) = -\, CS_X(\widehat{A})$ complete the proof of part (ii). □

6. Existence of Perturbation Curves

This section is devoted to finding loops in X with certain properties required for our transversality arguments in Section 3. The basic question is whether the trace of holonomy can detect a tangent vector to the flat moduli space. In terms of a one-parameter family A_t of irreducible flat $SU(3)$ connections, we ask: does there exist an element $\gamma \in \pi_1(X)$ such that

$$\tfrac{d}{dt}\, tr\, hol_\gamma(A_t)\big|_{t=0} \neq 0?$$

The answer is no if $A_t = g_t\, A_0$, so we must also assume that A_t is not tangent to the gauge orbit $\mathcal{G}A_0$. In fact, we need this for any path A_t of connections such that A_0 is flat and A_t is flat to first order (i.e., $\frac{d}{dt} F_{A_t}\big|_{t=0} = 0$). An affirmative answer to this question for $SU(2)$ and $SU(3)$ is given in the first two subsections. The last subsection treats the reducible case, where second order arguments are required.

6.1. **First order arguments.** To start, we introduce some notation. Given a flat connection A and a based loop $\ell : [0,1] \longrightarrow X$, let $H_\ell(A) \in SU(3)$ be the holonomy of A around ℓ. For $a \in \Omega^1(X, su(3))$, let $I_\ell(a, A) \in su(3)$ be the integral

$$I_\ell(a, A) = \int_0^1 P_\ell(0,t)^{-1} a_{\ell(t)} P_\ell(0,t)dt,$$

where $P_\ell(0,t)$ is the parallel translation from 0 to t along ℓ using the connection A. When A and a are clear from context, we write simply H_ℓ and I_ℓ.

If $A_t = A + ta + O(t^2)$, by Corollary 2.7, we see that $\frac{d}{dt}\, tr\, H_\ell(A_t)\big|_{t=0} = tr(H_\ell(A) I_\ell(a, A))$.

Proposition 6.1. *Suppose A is a flat $SU(3)$ connection. If $a \in \mathcal{H}^1_A(X; su(3))$ is non-zero, then there is a loop ℓ so that I_ℓ projects non-trivially onto $z(H_\ell)$, the Lie algebra of the centralizer $Z(H_\ell)$.*

Proof. Consider the differential equation $d_A\xi = a$. We can solve this equation locally on any 3-ball $B \subset X$ since $H^1_A(B; su(3)) = 0$. Because a is not exact, there is no global solution. Thus there exists some loop $\ell : [0,1] \longrightarrow X$ (which can be taken to be embedded) for which the local solutions do not match up at the ends. Hence

$a|_\ell$ is not exact, meaning that its decomposition $a|_\ell = a_h + d_A b$ into harmonic and exact parts has $a_h \neq 0$. From this point on, we restrict our attention to the pullback connection $\ell^*(A)$ on the $SU(3)$ bundle over the circle $S^1 = [0,1]/0 \sim 1$ pulled back via the loop ℓ.

Note that a_h is Hodge dual with respect to the metric on the loop to a covariantly constant 0-form, and so integrates to something nonzero in $\mathcal{H}^0_{\ell^*(A)}(S^1; su(3))$, the Lie algebra of $\mathrm{Stab}(\ell^*(A))$. By the fundamental theorem of calculus, the exact part integrates to $H_\ell^{-1} b_0 H_\ell - b_0$, where b_0 denotes the value of b at the basepoint. This latter $su(3)$ element is orthogonal to $\mathcal{H}^0_{\ell^*(A)}(S^1; su(3))$ (note that its left translation to $T_{H_\ell} SU(3)$ is tangent to the adjoint orbit of H_ℓ). □

The following 'warm-up' proposition treats the case $SU(2)$.

Proposition 6.2. *If A is an irreducible flat $SU(2)$ connection and $a \in \mathcal{H}^1_A(X; su(2))$ is nonzero, then there exists a curve γ with $tr(H_\gamma I_\gamma) \neq 0$.*

Proof. Since a is nonzero and harmonic, by Proposition 6.1 we can choose a curve ℓ so that $\Pi_{z(H_\ell)}(I_\ell) \neq 0$. Gauge transform A so that

$$H_\ell = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

is diagonal and write

$$I_\ell = \begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix}, \quad 0 \neq \alpha \in \mathbb{R}.$$

Then $tr(H_\ell I_\ell) = \alpha(\bar{\lambda} - \lambda) \neq 0$ unless $H_\ell = \pm I$. Taking $\gamma = \ell$ proves the claim if $H_\ell \neq \pm I$. Otherwise, we can always find γ so that $tr(H_\gamma I_\ell) \neq 0$. Using the fact that H_ℓ is central, it follows that

$$\begin{aligned} tr(H_{\gamma\cdot\ell} I_{\gamma\cdot\ell}) &= tr(H_\gamma H_\ell I_\ell) + tr(H_\gamma I_\gamma H_\ell) \\ &= \pm(tr(H_\gamma I_\ell) + tr(H_\gamma I_\gamma)). \end{aligned}$$

Since $tr(H_\gamma I_\ell)$ is nonzero, it follows that either $tr(H_{\gamma\cdot\ell} I_{\gamma\cdot\ell})$ or $tr(H_\gamma I_\gamma)$ is also nonzero, and this proves the proposition. □

The same is true for $SU(3)$, but it takes more work to prove.

Theorem 6.3. *If A is an irreducible flat $SU(3)$ connection and $a \in \mathcal{H}^1_A(X; su(3))$ is nonzero, then there exists a curve γ with $tr(H_\gamma I_\gamma) \neq 0$.*

Proof. Choose ℓ so that $\Pi_{z(H_\ell)}(I_\ell) \neq 0$. Gauge transform so that

$$H_\ell = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

is diagonal and write

$$I_\ell = \begin{pmatrix} i\alpha_1 & & * \\ & i\alpha_2 & \\ \bar{*} & & i\alpha_3 \end{pmatrix},$$

where α_i are real numbers, not all zero. Of course, $\lambda_3 = (\lambda_1\lambda_2)^{-1}$ and $\alpha_3 = -\alpha_1 - \alpha_2$.

If H_ℓ has only one eigenvalue, namely if $\lambda_1 = \lambda_2 = \lambda_3$, then H_ℓ is central and the theorem follows from the same argument as was used to prove Proposition 6.2. Otherwise, either H_ℓ has three distinct eigenvalues or it may be further conjugated so that $\lambda_1 = \lambda_2 \neq \lambda_3$. The following argument treats only the first of these two cases. The second case requires a more elaborate argument, given in the next subsection.

Assume λ_1, λ_2 and λ_3 are all distinct. Suppose first of all that $\alpha_i = 0$ for some i, which can be taken (wlog) to be $i = 3$. Since $tr(I_\ell) = 0$,

$$tr(H_\ell I_\ell) = \lambda_1(i\alpha_1) + \lambda_2(i\alpha_2) = i\alpha_1(\lambda_1 - \lambda_2),$$

which is nonzero since $\alpha_1 \neq 0$ and $\lambda_1 \neq \lambda_2$.

Now suppose $\alpha_i \neq 0$ for all i. By replacing a with $-a$, if necessary, we can assume that two of the α_i's are positive, which we take (wlog) to be α_1 and α_2. Then

$$tr(H_\ell I_\ell) = i\lambda_1\alpha_1 + i\lambda_2\alpha_2 - i(\lambda_1\lambda_2)^{-1}(\alpha_1 + \alpha_2).$$

Thus $tr(H_\ell I_\ell) = 0$ implies $\lambda_1\alpha_1 + \lambda_2\alpha_2 = (\lambda_1\lambda_2)^{-1}(\alpha_1 + \alpha_2)$. If this were the case, then $|\lambda_1\alpha_1 + \lambda_2\alpha_2| = |\alpha_1 + \alpha_2|$, which is only possible if $\lambda_1 = \lambda_2$, a contradiction.

The following example illustrates the difficulties when H_ℓ has only two distinct eigenvalues. Suppose

$$H_\ell = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda^{-2} \end{pmatrix}.$$

Then one see that $tr(H_\ell I_\ell) = 0$ if

$$I_\ell = \begin{pmatrix} i\alpha & & * \\ & -i\alpha & \\ \bar{*} & & 0 \end{pmatrix}.$$

The next subsection is devoted to treating this problematic case. Observe that we can assume that H_ℓ has infinite order for the following reason. If H_ℓ has finite order k and if γ is chosen so that $tr(H_\gamma I_{\ell^k}) \neq 0$, then just as in the proof of Proposition 6.2, we compute that

$$tr(H_{\gamma\cdot\ell^k} I_{\gamma\cdot\ell^k}) = tr(H_\gamma I_{\ell^k}) + tr(H_\gamma I_\gamma).$$

But $tr(H_\gamma I_{\ell^k}) \neq 0$, hence it follows that one of the other two terms is also non-zero.

6.2. **Linear algebra.** In this subsection, we complete the proof of Theorem 6.3 demonstrating the existence of perturbation curves with certain properties. The remaining case is when H_ℓ has only two distinct eigenvalues. As indicated in the previous subsection, we can further assume that H_ℓ has infinite order. Although H_ℓ may not have three distinct eigenvalues, the following proposition assures us that H_γ has three distinct eigenvalues for some loop γ.

Proposition 6.4. *If $\varrho : \pi_1(X) \longrightarrow SU(3)$ is an irreducible representation, then there exists some element $\gamma \in \pi_1(X)$ such that $\varrho(\gamma)$ has three distinct eigenvalues.*

Remark. Besides the existence of the irreducible, rank three representation, the proof makes no assumptions on the group $\pi_1(X)$.

Proof. By irreducibility, we can find ℓ with $\varrho(\ell)$ noncentral. Set $L = \varrho(\ell)$. Obviously, we are done unless L has only two distinct eigenvalues. Since the conclusion of the proposition is invariant under conjugation, we can assume

$$L = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda^{-2} \end{pmatrix}$$

is diagonal. By irreducibility of ϱ, there exists $m \in \pi_1(X)$ so that $\varrho(m)$ does not commute with L. Set $M = \varrho(m)$. Thus neither M nor LM is diagonal. Of course, we can also assume that M has only two distinct eigenvalues; otherwise we are done! Let μ be the eigenvalue of M of multiplicity two. Now both L and M have 2-dimensional eigenspaces, so for dimensional reasons, L and M have a common eigenvector. After conjugating by a matrix commuting with L, it follows that M can be written in block diagonal form:

$$M = \begin{pmatrix} \mu & 0 \\ 0 & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-2} \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$$

and $|a|^2 + |b|^2 = 1$. The matrix product LM also comes in block form:

$$LM = \begin{pmatrix} \lambda\mu & 0 \\ 0 & B \end{pmatrix} \qquad \text{where} \qquad B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-2} \end{pmatrix} A.$$

We claim that LM has three distinct eigenvalues. First of all, notice that the two eigenvalues of B are distinct; otherwise LM would be diagonal, in which case L and M would commute. So it suffices to prove that $\lambda\mu$ is not an eigenvalue of B.

Suppose to the contrary that $\lambda\mu$ is an eigenvalue of B, i.e., suppose $(\lambda\mu)^2 - tr(B)(\lambda\mu) + \det(B) = 0$. Computing $tr(B)$ directly, one finds

$$tr(B) = |a|^2(\lambda\mu + \lambda^{-2}\mu^{-2}) + (1 - |a|^2)(\lambda\mu^{-2} + \lambda^{-2}\mu).$$

Plugging this into the characteristic equation and using $\det B = \lambda^{-1}\mu^{-1}$ gives

$$(1-|a|^2)(\lambda^2\mu^2 + \lambda^{-1}\mu^{-1} - \lambda^2\mu^{-1} - \lambda^{-1}\mu^2) = 0.$$

So either $|a| = 1$, implying $A = \pm I$ and contradicting our choice of M, or, after multiplying by $\lambda\mu$,

$$0 = \lambda^3\mu^3 + 1 - \lambda^3 - \mu^3 = (\lambda^3 - 1)(\mu^3 - 1).$$

However, $\lambda^3 = 1$ implies L is central and $\mu^3 = 1$ implies M is central, each giving contradictions. Hence $\lambda\mu$ is not an eigenvalue of B, which proves our claim. □

With regard to Theorem 6.3, we have already proved the existence of γ unless $H_\ell(A)$ has two distinct eigenvalues, so assume

$$H_\ell(A) = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda^{-2} \end{pmatrix}.$$

Set $L_t = H_\ell(A_t)$ where $A_t = A + ta$. Observe moreover that we are done unless

$$\left.\frac{dL_t}{dt}\right|_{t=0} = L_0 \begin{pmatrix} i\alpha & & 0 \\ & -i\alpha & \\ 0 & & 0 \end{pmatrix},$$

where $\alpha \neq 0$.

Before stating the next lemma, we make a definition.

Definition 6.5. *For a fixed angle $\eta \in [0, 2\pi)$, let G_η be the subset of $SU(3)$ consisting of matrices of the form*

$$M = \begin{pmatrix} a & be^{-i\eta} & c \\ be^{i\eta} & a & ce^{i\eta} \\ c' & c'e^{-i\eta} & d \end{pmatrix}$$

for $a, b, c, c', d \in \mathbb{C}$. Note that $M \in SU(3) \Rightarrow |c| = |c'|$.

Clearly G_η is a subgroup; in fact, matrices of this form in $SL(3,\mathbb{C})$ form a subgroup of complex codimension 4, so one would expect that G_η has real codimension 4 in $SU(3)$. This is indeed the case; if M is chosen as above and

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ e^{i\eta} & 0 & -e^{i\eta} \\ 0 & \sqrt{2} & 0 \end{pmatrix},$$

then $P \in U(3)$,

$$P^{-1}MP = \begin{pmatrix} a+b & \sqrt{2}c & 0 \\ \sqrt{2}c' & d & 0 \\ 0 & 0 & a-b \end{pmatrix},$$

and so G_η is conjugate to the subgroup $S(U(2) \times U(1))$.

Lemma 6.6. *Suppose that $L_t, M_t : (-\epsilon, \epsilon) \longrightarrow SU(3)$ are smooth paths. Write $L_0' = L_0^{-1} \left.\frac{dL_t}{dt}\right|_{t=0}$, and assume both L_0 and L_0' are diagonal, with*

$$L_0 = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda^{-2} \end{pmatrix} \text{ and } L_0' = \begin{pmatrix} i\alpha & & 0 \\ & -i\alpha & \\ 0 & & 0 \end{pmatrix}$$

for λ a complex unit of infinite order and for $\alpha \neq 0$. If

$$\left.\tfrac{d}{dt}\, tr(W_t)\right|_{t=0} = 0$$

for every word W_t in L_t and M_t, then $M_0 \in G_\eta$ for some η.

Proof. For general $L, M \in SU(3)$ with

$$L = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

diagonal and $M = (\mu_{ij})$ arbitrary, it is not difficult to verify that

$$
\begin{aligned}
tr(LM) &= \sum_{i=1}^{3} \lambda_i \mu_{ii}, \\
tr(LML^{-1}M^{-1}) &= \sum_{i,j=1}^{3} \lambda_i \bar{\lambda}_j |\mu_{ij}|^2.
\end{aligned}
$$

Now suppose L_t, M_t are as in the hypotheses. We write $M_t = (\mu_{ij}(t))$ and let $\mu_{ij} = \mu_{ij}(0)$ for convenience. Applying the above formula to $L_t^k M_t$ and $L_t^k M_t L_t^{-k} M_t^{-1}$ and taking derivatives, we see from the hypotheses that

$$
\begin{aligned}
0 &= \tfrac{d}{dt}\, tr(L_t^k M_t)\big|_{t=0} \\
&= \lambda^k \tfrac{d}{dt}\left(\mu_{11}(t) + \mu_{22}(t)\right)\big|_{t=0} + \bar{\lambda}^{2k} \tfrac{d}{dt}\mu_{33}(t)\big|_{t=0} + ik\alpha\lambda^k\left(\mu_{11} - \mu_{22}\right),
\end{aligned}
$$

and that

$$
\begin{aligned}
0 &= \tfrac{d}{dt}\, tr(L_t^k M_t L_t^{-k} M_t^{-1})\big|_{t=0} \\
&= \tfrac{d}{dt}\left(|\mu_{11}(t)|^2 + |\mu_{12}(t)|^2 + |\mu_{21}(t)|^2 + |\mu_{22}(t)|^2 + |\mu_{33}(t)|^2\right)\big|_{t=0} \\
&\quad + \lambda^{3k} \tfrac{d}{dt}\left(|\mu_{13}(t)|^2 + |\mu_{23}(t)|^2\right)\big|_{t=0} + \bar{\lambda}^{3k} \tfrac{d}{dt}\left(|\mu_{31}(t)|^2 + |\mu_{32}(t)|^2\right)\big|_{t=0} \\
&\quad + k\alpha\left\{2(|\mu_{12}|^2 - |\mu_{21}|^2) + \lambda^{3k}(|\mu_{13}|^2 - |\mu_{23}|^2) - \bar{\lambda}^{3k}(|\mu_{31}|^2 - |\mu_{32}|^2)\right\}.
\end{aligned}
$$

Since both equations hold for all $k \geq 0$ and since λ has infinite order, we deduce that:

$$
\begin{array}{ll}
\text{(i)}\ \ \mu_{11} = \mu_{22}, & \text{(ii)}\ \ |\mu_{12}| = |\mu_{21}|, \\
\text{(iii)}\ \ |\mu_{13}| = |\mu_{23}|, & \text{(iv)}\ \ |\mu_{31}| = |\mu_{32}|.
\end{array}
$$

Here, (i) is a consequence of the first equation and (ii)–(iv) come from the second. The last three conditions are equivalent to the existence of angles η_1, η_2, and η_3 with $\mu_{21} = e^{i2\eta_1}\mu_{12}$, $\mu_{23} = e^{i\eta_2}\mu_{13}$, and $\mu_{32} = e^{-i\eta_3}\mu_{31}$.

To conclude that $M_0 \in G_\eta$, we just need to show that $\eta_1 = \eta_2 = \eta_3 \mod (2\pi)$. Applying (i) to $(M_0)^2$ implies $\mu_{13}\mu_{31} = \mu_{23}\mu_{32}$, thus $\eta_2 = \eta_3 \mod (2\pi)$. Now apply the unitary condition to M_0 to see $0 = \sum_{j=1}^{3} \mu_{ij}\bar{\mu}_{3j}$ for $i = 1, 2$. Comparing these, we conclude $\eta_1 = \eta_2 \mod (2\pi)$. This completes the proof of the lemma. □

To establish Theorem 6.3, we seek a curve γ such that $tr(H_\gamma I_\gamma) \neq 0$. Setting $A_t = A + ta$, this is equivalent to the condition that $\frac{d}{dt}\, tr\, H_\gamma(A_t)\big|_{t=0} \neq 0$. According to the previous lemma, letting γ range over all words in L_0 and M_0, the only way this can fail is if $M_0 \in G_\eta$ for some η. We shall show in the following argument that the irreducibility of A guarantees the existence of an $M = H_m(A)$ such that $M \notin G_\eta$ for any η.

Proof of Theorem 6.3. We provide the proof in the remaining case when $L_0 = H_\ell(A)$ has two distinct eigenvalues and is of infinite order. Set $A_t = A + ta$ and $L_t = H_\ell(A_t)$. By Proposition 6.4, we have a loop m_1 such that $H_{m_1}(A)$ has three distinct eigenvalues. Set $M_1 = H_{m_1}(A)$ and $M_{1,t} = H_{m_1}(A_t)$. Assume first that $M_1 \notin G_\eta$ for any $\eta \in [0, 2\pi]$. By Lemma 6.6, there is a word W_t in L_t and $M_{1,t}$ such that $\frac{d}{dt}\, tr\, W_t\big|_{t=0} \neq 0$. Taking γ as the loop obtained from the corresponding word in ℓ and m_1, then $W_t = H_\gamma(A_t)$ and hence $\frac{d}{dt}\, tr\, H_\gamma(A_t)\big|_{t=0} \neq 0$, which proves the theorem in this case.

So now suppose $M_1 \in G_{\eta_1}$ and write

$$M_1 = \begin{pmatrix} a_1 & b_1 e^{-i\eta_1} & c_1 \\ b_1 e^{i\eta_1} & a_1 & c_1 e^{i\eta_1} \\ c_1' & c_1' e^{-i\eta_1} & d_1 \end{pmatrix}.$$

Although A_t has been gauge transformed so that the path $H_\ell(A_t)$ is diagonal, we can further conjugate by a diagonal matrix since it acts trivially on $H_\ell(A_t)$. Applying such a conjugation to M_1, we can arrange that b_1 and c_1 are both real and non-negative.

Since A is irreducible, we can choose $m_2 \in \pi_1(X)$ such that $M_2 = H_{m_2}(A) \notin G_{\eta_1}$. Repeating the argument above with M_1 replaced by M_2, we can assume that $M_2 \in G_{\eta_2}$ for some η_2 and write

$$M_2 = \begin{pmatrix} a_2 & b_2 e^{-i\eta_2} & c_2 \\ b_2 e^{i\eta_2} & a_2 & c_2 e^{i\eta_2} \\ c_2' & c_2' e^{-i\eta_2} & d_2 \end{pmatrix}.$$

We now claim that one of $M_1 M_2$ and $M_1^{-1} M_2$ is not contained in G_η for any η. This completes the proof of Theorem 6.3 by repeating once again the above argument with M_1 replaced by either M_1M_2 or $M_1^{-1}M_2$ and invoking Lemma 6.6 to produce the curve γ with the desired properties.

So, it only remains to establish the claim, which is proved by contradiction. Suppose that M_1M_2 is contained in G_η for some η. Equating the $(1,1)$ and $(2,2)$ entries of M_1M_2, we find

$$b_1b_2u^2 + c_1c_2'u - (b_1b_2 + c_1c_2') = 0, \tag{14}$$

where $u = e^{i(\eta_2-\eta_1)}$.

Suppose first of all that $b_1 = 0$. Because $c_1 = 0 \Rightarrow M_1$ is diagonal, and $u = 1 \Rightarrow M_2 \in G_{\eta_1}$, neither of which is the case, the only possibility is that $c_2' = 0$. But writing out M_1M_2 and demanding that the off-diagonal terms have the required form, this would imply that $b_2 = 0$ or $u = 1$, both of which lead to contradictions.

So assume $b_1 \neq 0$. By similar considerations, we can also assume $b_2 \neq 0$. Solving equation (14) for u gives

$$u = -1 - \frac{c_1c_2'}{b_1b_2},$$

since we have already seen that the other possibility, namely $u = 1$, leads to a contradiction.

The same reasoning applied to $M_1^{-1}M_2$ shows that

$$u = -1 - \frac{\overline{c}_1'c_2'}{\overline{b}_1b_2}.$$

Equating these two formulas for u gives

$$c_1\overline{b}_1 = \overline{c}_1'b_1.$$

Since b_1 and c_1 are real, this shows that $\overline{c}_1' = c_1 = c_1'$, which forces both a_1 and d_1 to also be real. It immediately follows that $M_1^{-1} = M_1^* = M_1$, hence the eigenvalues of M_1 equal ± 1. In particular, M_1 has at most two distinct eigenvalues, which contradicts our choice of M_1. This proves the claim and concludes the proof of Theorem 6.3. □

Given loops $\ell_1, \dots, \ell_n$ in X, define gauge invariant functions $f_j, g_j : \mathcal{A} \longrightarrow \mathbb{R}$ for $j = 1, \dots, n$ to be the real and imaginary parts of $tr\, hol_{\ell_j}(A)$, so that

$$tr(hol_{\ell_j}(A)) = f_j(A) + ig_j(A).$$

Note that if A is an $SU(2)$ connection, then $g_j(A) = 0$.

Corollary 6.7. (i) *If A is an irreducible, flat $SU(2)$ connection, then there exist loops $\ell_1, \dots, \ell_n$ so that the map from $\mathcal{H}^1_A(X; su(2))$ to $\mathbb{R}^n$ given by $a \mapsto (Df_1(A)(a), \dots, Df_n(A)(a))$ is injective.*

(ii) *If A is an irreducible, flat $SU(3)$ connection, then there exist loops $\ell_1, \dots, \ell_n$ so that the map from $\mathcal{H}^1_A(X; su(3))$ to $\mathbb{R}^{2n}$ given by $a \mapsto (Df_1(A)(a), Dg_1(A)(a), \dots, Df_n(A)(a), Dg_n(A)(a))$ is injective.*

6.3. **Second order arguments.** Suppose now that A is a reducible flat $SU(3)$ connection on X. Then part (i) of Corollary 6.7 allows us to find loops about which the real part of the derivative of trace of holonomy detects any first order deformations of A in directions tangent to the reducible stratum, i.e., in the directions of $\mathcal{H}^1_A(X; \mathfrak{h})$. But invariance under the gauge group, in particular under $\text{Stab}(A) = U(1)$, prevents the derivative from detecting first order deformations in directions normal to the reducible stratum, i.e., in the directions of $\mathcal{H}^1_A(X; \mathfrak{h}^\perp)$. Instead, we consider second derivatives of the gauge invariant functions in these directions. This portion of the argument closely parallels the argument used to handle abelian flat connections in the $SU(2)$ moduli space [9].

Notice first that $\mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ is a module over the quaternions $\mathbb{H}$. To see this, let $SP(1)$ be the unit quaternions and define $\phi : SU(2) \to SP(1)$ by

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto a + Jb$$

and $F : \mathbb{C}^2 \to \mathbb{H}$ by $F(v_1, v_2) = v_1 + Jv_2$. Then for $A \in SU(2)$ and $v \in \mathbb{C}^2$,

$$F(Av) = \phi(A)F(v).$$

This turns the action of $SU(2)$ on $\mathbb{C}^2$ into left multiplication by elements of $SP(1)$ on $\mathbb{H}$.

Now suppose $\varrho : \pi_1(X) \to SU(2)$ is an irreducible representation and let E_ϱ be the flat bundle $\widetilde{X} \times_{\pi_1(X)} \mathbb{C}^2$, where $\widetilde{X}$ is the universal cover of X and $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and via the canonical representation of ϱ on $\mathbb{C}^2$. We identify E_ϱ as a flat bundle with the subbundle of $\operatorname{ad} P = X \times su(3)$ corresponding to $\mathfrak{h}^\perp \subset su(3)$. The de Rham theorem provides an isomorphism $\mathcal{H}^1_A(X;\mathfrak{h}^\perp) \cong H^1(X;E_\varrho)$. Here, $H^1(X;E_\varrho) = Z^1(X;E_\varrho)/B^1(X;E_\varrho)$ is by definition the space of 1-cocycles modulo the 1-coboundaries. Using a presentation $\pi_1(X) = \langle x_1,\ldots,x_n \mid r_1,\ldots,r_m\rangle$, we can identify the 1-cochains as elements $(v_1,\ldots,v_n) \in \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \cong \mathbb{H}^n$ and the subspaces $Z^1(X;E_\varrho)$ of 1-cocycles and $B^1(X;E_\varrho)$ of 1-coboundaries as submodules. For example, $(v_1,\ldots,v_n)$ is a coboundary if and only if there is some $v \in \mathbb{C}^2$ such that $v_i = v - \varrho(x_i)v$ for $i = 1,\ldots,n$. Observe that $B^1(X;E_\varrho)$ is closed under right multiplication by elements in $\mathbb{H}$. Similarly, $(v_1,\ldots,v_n)$ is a cocycle if and only if the following linear equations, which are derived from the relations $r_1,\ldots,r_m$ using the Fox differential calculus, are satisfied:

$$\begin{aligned} M_{11}v_1 + \cdots + M_{1n}v_n &= 0 \\ &\vdots \\ M_{m1}v_1 + \cdots + M_{mn}v_n &= 0. \end{aligned} \tag{15}$$

Here each M_{ij} is a sum of $SU(2)$ matrices and thus is a 2×2 matrix of the form

$$M_{ij} = \begin{pmatrix} a_{ij} & -\bar{b}_{ij} \\ b_{ij} & \bar{a}_{ij} \end{pmatrix}$$

for some $a_{ij}, b_{ij} \in \mathbb{C}$. For $v = (v_1,v_2) \in \mathbb{C}^2$ and $h = z_1 + Jz_2 \in \mathbb{H}$, where $z_1,z_2 \in \mathbb{C}$, set $v \cdot h = (z_1v_1 - z_2\bar{v}_2, z_2\bar{v}_1 + z_1v_2) \in \mathbb{C}^2$. (This is just multiplication in $\mathbb{H}$ under the isomorphism $F : \mathbb{C}^2 \cong \mathbb{H}$.) Now if $(v_1,\ldots,v_n)$ satisfies (15) above, then so does $(v_1\cdot h,\ldots,v_n\cdot h)$. This shows that $Z^1(X;E_\varrho)$ is closed under right multiplication by elements of $\mathbb{H}$. Since both $B^1(X;E_\varrho)$ and $Z^1(X;E_\varrho)$ are right $\mathbb{H}$-modules, so is $H^1(X;E_\varrho) = Z^1(X;E_\varrho)/B^1(X;E_\varrho)$.

Recalling the notation of Definition 3.2, we use $\operatorname{Herm}\mathcal{H}^1_A(X;\mathfrak{h}^\perp)$ to denote the set of all symmetric $\operatorname{Stab}(A) \cong U(1)$ invariant bilinear forms on $\mathcal{H}^1_A(X;\mathfrak{h}^\perp)$, regarded as a real vector space with a $U(1)$ action.

Proposition 6.8. *If A is a reducible flat connection, then there exist loops $\ell_1, \dots, \ell_{n_1}$ in X and a set $F = \{f_1, \dots, f_n\}$ of gauge invariant functions such that:*

(i) *Each $f_i \in F$ is the real or imaginary part of tr hol_{ℓ_j} for some $j = 1, \dots, n_1$.*

(ii) *The map $\mathbb{R}^n \longrightarrow \operatorname{Herm} \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ given by $(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i \operatorname{Hess} f_i(A)$ is surjective.*

(iii) *$Df_i(A) = 0$ for $i = 1, \dots, n$.*

Proof. Assume A has been gauge transformed to take values in $su(2) \subset su(3)$ and denote by $\widehat{A}$ the associated irreducible $SU(2)$ connection. In order to construct the loops $\ell_1, \dots, \ell_{n_1}$, we will need to introduce curves in X that are in a certain sense dual to a basis for $\mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ over $\mathbb{H}$.

Let $\varrho : \pi_1(X) \to SU(2)$ be the irreducible $SU(2)$ representation associated to $\widehat{A}$, and let $E_\varrho = \widetilde{X} \times_{\pi_1(X)} \mathbb{C}^2$ as before. Consider $H_i(X; E_\varrho)$, homology with local coefficients in E_ϱ, which is by definition the homology of the complex

$$\cdots \longrightarrow C_i(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{C}^2 \xrightarrow{\partial_i \otimes 1} C_{i-1}(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{C}^2 \longrightarrow \cdots .$$

From our previous discussion, it is not hard to see that $H_1(X; E_\varrho)$ is a right $\mathbb{H}$-module. Thus, we have a basis for $H_1(X; E_\varrho)$ over $\mathbb{H}$ consisting of classes each of which can be represented by a $\mathbb{C}^2$-labelled curve $\widetilde{\gamma}_i$ in the universal cover $\widetilde{X}$ of X. Each $\widetilde{\gamma}_i$ is a lift of a loop γ_i in X with $hol_{\gamma_i}(A) = 1$ (because the labelled lift of γ_i lies in $\ker \partial_1 \otimes 1$ if and only if the holonomy of A around γ_i is trivial). Let $\omega_1, \dots, \omega_m$ be the Hom dual basis for $H^1(X; E_\varrho)$ over $\mathbb{H}$. Of course, each ω_i determines a real, 4-dimensional subspace $V_i = \phi(\operatorname{span}_{\mathbb{H}} \omega_i) \subset \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ where $\phi : H^1(X; E_\varrho) \to \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ is the isomorphism provided by the de Rham theorem. Each V_i is preserved by the subgroup $\operatorname{Stab}(A) \subset \mathcal{G}$, thus

$$\mathcal{H}^1_A(X; \mathfrak{h}^\perp) = V_1 \oplus \cdots \oplus V_m \tag{16}$$

is a decomposition of $\mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ into 2-dimensional complex vector spaces. We denote by a_i the image of $a \in \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ under the projection $p_i : \mathcal{H}^1_A(X; \mathfrak{h}^\perp) \to V_i$.

Let $\mathcal{U} = \operatorname{Herm} \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$ be the space of symmetric $\operatorname{Stab}(A) = U(1)$ invariant bilinear forms on $\mathcal{H}^1_A(X; \mathfrak{h}^\perp)$. Our goal is to find a collection of loops such that the

Hessians of the real and imaginary parts of the trace of holonomy functions around these loops span $\mathcal{U}$.

There is a decomposition of $\mathcal{U}$ corresponding to (16) given by $\mathcal{U} = \bigoplus_{i\leq j} \mathcal{U}_{ij}$, where $B \in \mathcal{U}_{ij}$ in case

$$B(a,b) = \begin{cases} B(a_i, b_i) & \text{if } i = j, \\ B(a_i, b_j) + B(a_j, b_i) & \text{if } i \neq j. \end{cases}$$

Thus every $B \in \mathcal{U}_{ij}$ is entirely determined by its restriction to $V_i \times V_j$. Let $\{a, b\}$ be a basis for V_i and $\{c, d\}$ a basis for V_j. In terms of the real bases $\{a, ia, b, ib\}$ for V_i and $\{c, ic, d, id\}$ for V_j, the restriction of B to $V_i \times V_j$ is a real 4×4 matrix of the form

$$\begin{bmatrix} x & 0 & y & z \\ 0 & x & -z & y \\ y & -z & w & 0 \\ z & z & 0 & w \end{bmatrix} \quad \text{if } i = j, \text{ and } \quad \begin{bmatrix} p & q & r & s \\ -q & p & -s & r \\ t & -u & v & w \\ u & t & -w & v \end{bmatrix} \text{ if } i \neq j. \tag{17}$$

From this, it follows that

$$\dim \mathcal{U}_{ij} = \begin{cases} 4 & \text{if } i = j \\ 8 & \text{if } i \neq j. \end{cases}$$

We prove the proposition by constructing, for each $i \leq j$, gauge invariant functions satisfying conditions (i) and (iii) such that their Hessians at A span $\mathcal{U}_{ij}$. We begin with the case $i = j$.

Given $\beta : [0, 1] \to X$ with $\beta(0) = \beta(1)$, parallel translation can be used to associate a function $\alpha : [0, 1] \to \mathfrak{h}^\perp$ to any $su(3)$-valued 1-form a by setting

$$\alpha(t)dt = P_\beta(0, t)^{-1} a_{\beta(t)} P_\beta(0, t), \tag{18}$$

where $P_\beta(0, t)$ is parallel translation by A along β from $\beta(0)$ to $\beta(t)$. If $a \in \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$, then $\int_0^1 \alpha(t)dt = \int_{\gamma_i} a \in \mathfrak{h}^\perp$. The linear transformation $\mathcal{H}^1_A(X; \mathfrak{h}^\perp) \to \mathfrak{h}^\perp$ defined by $a \mapsto \int_{\gamma_i} a$ has kernel $V_1 \oplus \cdots \widehat{V_i} \cdots \oplus V_m$ (because the basis $\omega_1, \ldots, \omega_m$ is Hom dual to $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_m$) and determines an isomorphism $V_i \to \mathfrak{h}^\perp$.

Note how the correspondence (18) behaves for products of loops. If $\beta = \ell_1 \cdots \ell_k : [0,k] \to X$ (where each $\ell_i : [i-1,i] \to X$ is a loop), define $\alpha_i : [i-1,i] \to \mathfrak{h}^\perp$ by $\alpha_i(t)dt = P_{\ell_i}(i-1,t)^{-1} a_{\ell_i(t)} P_{\ell_i}(i-1,t)$. Defining $\alpha : [0,k] \to \mathfrak{h}^\perp$ by (18), then

$$\alpha(t) = hol_{\ell_1}(A)^{-1} \cdots hol_{\ell_{i-1}}(A)^{-1} \alpha_i(t)\; hol_{\ell_{i-1}}(A) \cdots hol_{\ell_1}(A) \tag{19}$$

for $t \in [i-1,i]$.

Lemma 6.9. *Suppose ℓ is a loop with $L := hol_\ell(A) \in SU(3)$ nontrivial. If $a, b \in \mathcal{H}^1_A(X;\mathfrak{h}^\perp)$ and if we set $\xi_i = \int_{\gamma_i} a \in \mathfrak{h}^\perp$ and $\zeta_i = \int_{\gamma_i} b \in \mathfrak{h}^\perp$, then*

(i) Hess $tr\, hol_{\gamma_i}(A)(a,b) = 2\, tr(\xi_i \zeta_i)$.

(ii) Hess $tr\, hol_{\ell\cdot\gamma_i}(A)(a,b) = tr(L(\xi_i\zeta_i + \zeta_i\xi_i))$.

Proof. Let $B, B_\ell : \mathcal{H}^1_A(X;\mathfrak{h}^\perp) \times \mathcal{H}^1_A(X;\mathfrak{h}^\perp) \to \mathbb{C}$ be the symmetric, bilinear pairings coming from the Hessians at A of $tr\, hol_{\gamma_i}$ and $tr\, hol_{\ell\cdot\gamma_i}$, respectively. (Notice that $B(a,b) \in \mathbb{R}$ because $hol_{\gamma_i}(A)$ is trivial. This follows from 2.7 (ii) and the elementary fact that $tr(\xi\zeta) \in \mathbb{R}$ for $\xi, \zeta \in \mathfrak{h}^\perp$.)

Since B and B_ℓ are both symmetric, and since $\int_{\gamma_i}(a+b) = \xi_i + \zeta_i$, it suffices to show (i) and (ii) in the case $a = b$. To prove (i), parameterize γ_i by the interval $[0,1]$ and define $\alpha : [0,1] \to \mathfrak{h}^\perp$ associated to the 1-form $a \in \mathcal{H}^1_A(X;\mathfrak{h}^\perp)$ using (18). Using the formula from Corollary 2.7 (ii) and noting that $hol_{\gamma_i}(A)$ is trivial, it follows that

$$\begin{aligned}
\text{Hess}\, tr\, hol_{\gamma_i}(A)(a,a) &= \int_0^1 \int_0^s tr(\alpha(s)\alpha(t) + \alpha(t)\alpha(s))dtds \\
&= \int_0^1 \int_0^1 tr(\alpha(s)\alpha(t))dtds \\
&= tr\left(\int_0^1 \alpha(s)ds \int_0^1 \alpha(t)dt \right) = tr(\xi_i^2).
\end{aligned}$$

This proves (i).

To prove (ii), set $\beta = \ell \cdot \gamma_i$ and parameterize it by the interval $[0,2]$ so that the subintervals $[0,1]$ and $[1,2]$ parameterize ℓ and γ_i, respectively. Define $\alpha : [0,2] \to \mathbb{C}^2$ associated to the 1-form a using (18). Notice that $\int_0^1 \alpha(t)dt = 0$ because the restriction of any element of $\mathcal{H}^1_A(X;\mathfrak{h}^\perp)$ to a loop $\ell : S^1 \to X$ is exact whenever

$hol_\ell(A)$ is nontrivial (since $\mathcal{H}^0_{\ell^*(A)}(S^1;\mathfrak{h}^\perp)=0$, which implies that $\mathcal{H}^1_{\ell^*(A)}(S^1;\mathfrak{h}^\perp)=0$ by Poincaré duality). Hence by (19) we see that

$$\int_0^2 \alpha(t)dt = \int_1^2 \alpha(t)dt = L^{-1}\xi_i L.$$

Appealing once again to Corollary 2.7 (ii), it follows that

$$\begin{aligned}\text{Hess}\, tr\, hol_{\ell\cdot\gamma_i}(A)(a,a) &= \int_0^2\int_0^s tr\left[L(\alpha(s)\alpha(t)+\alpha(s)\alpha(t))\right] dtds \\ &= \int_0^2\int_0^2 tr\left[L\,\alpha(s)\alpha(t)\right] dsdt \\ &= tr\left(\int_0^2 L\alpha(s)ds\int_0^2\alpha(t)dt\right) = tr(\xi_i^2 L).\end{aligned}$$

□

Since $a_i = 0 \Rightarrow \int_{\gamma_i} a = \xi_i = 0$, it follows from (i) and (ii) above that the Hessians at A of the real and imaginary parts of $tr\, hol_{\gamma_i}$ and $tr\, hol_{\ell\cdot\gamma_i}$ lie in $\mathcal{U}_{ii}$. Consider the gauge invariant functions $f = \mathfrak{Re}\, tr\, hol_{\gamma_i}$ and $g_\ell = \mathfrak{Im}\, tr\, hol_{\ell\cdot\gamma_i}$, where $\mathfrak{Re}$ and $\mathfrak{Im}$ denote the real and imaginary parts. Note that f and g_ℓ obviously satisfy condition (i) of Proposition 6.8. Moreover, since $hol_{\gamma_i}(A)$ is trivial, $Df(A)=0$. This follows from formula (i) of Corollary 2.7. The same formula also implies that the imaginary part of $D\, tr\, hol_{\ell\cdot\gamma_i}(A)$ vanishes since $tr(L\xi)$ is real for $\xi\in\mathfrak{h}^\perp$ whenever the $SU(3)$ matrix L is in the image of the standard inclusion $SU(2)\to SU(3)$. This shows that $Dg_\ell(A)=0$, hence f and g_ℓ satisfy condition (iii) of Proposition 6.8. So, we only need to prove that we can span $\mathcal{U}_{ii}$ with the Hessians of such functions.

For this, we shall use the isomorphism $V_i \to \mathfrak{h}^\perp$ given by $a \mapsto \int_{\gamma_i} a$, along with the standard identification $\varphi : \mathfrak{h}^\perp \to \mathbb{C}^2$, to translate it into a question about symmetric, bilinear pairings $\mathbb{C}^2\times\mathbb{C}^2\to\mathbb{R}$. Denote by $\langle\cdot,\cdot\rangle$ the standard complex inner product on $\mathbb{C}^2$. If $\xi,\zeta\in\mathfrak{h}^\perp$ and $v,w\in\mathbb{C}^2$ are given by $v=\varphi(\xi)$ and $w=\varphi(\zeta)$, then

$$tr(\xi\zeta) = -2\mathfrak{Re}\langle v,w\rangle.$$

Moreover, if $\hat{L} = \left(\begin{smallmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{smallmatrix}\right)\in SU(2)$ and $L=\hat{L}\oplus 1\in SU(3)$, then

$$tr(L(\xi\zeta+\zeta\xi)) = -\langle\hat{L}(v),w\rangle - \langle\hat{L}(w),v\rangle - 2\mathfrak{Re}\langle v,w\rangle.$$

In terms of the real basis $\left\{\binom{1}{0}, \binom{i}{0}, \binom{0}{1}, \binom{0}{i}\right\}$ for $\mathbb{C}^2$, the symmetric bilinear form $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by

$$(v, w) \mapsto \langle \hat{L}(v), w \rangle + \langle \hat{L}(w), v \rangle + 2\mathfrak{Re}\langle v, w \rangle$$

has imaginary part represented by the matrix

$$\Psi(\hat{L}) = 2 \begin{bmatrix} s & 0 & -u & t \\ 0 & s & -t & -u \\ -u & -t & -s & 0 \\ t & -u & 0 & -s \end{bmatrix}. \tag{20}$$

where $\alpha = r + is$ and $\beta = t + iu$.

Now A is reducible (but *not* abelian) and thus we have $x, y \in \pi_1(X)$ such that $\varrho(x)$ and $\varrho(y)$ do not commute. We claim that the Hessians at A of the four functions

$$f, g_x, g_y, g_{xy}$$

derived from γ_i are linearly independent and form a basis for the 4-dimensional subspace $\mathcal{U}_{ii} \subset \mathcal{U}$.

To see this, restrict each Hessian to $V_i \times V_i$ and consider the associated symmetric 4×4 matrix of the form (17). For example, the matrix associated to Hess $f(A)$ equals -2 times the identity matrix. Clearly the image of $SU(2)$ under Ψ in (20) is the complementary subspace of dimension 3. Thus, it suffices to prove that the Hessians at A of g_x, g_y and g_{xy}, are linearly independent. One can see this by direct computation; arranging that $\varrho(x)$ is diagonal (by conjugation) and $\varrho(y)$ is not (by hypothesis), it becomes a routine exercise in linear algebra.

This proves that the Hessians at A of f, g_x, g_y and g_{xy} form a basis for $\mathcal{U}_{ii}$, and to conclude the proof of Proposition 6.8, we need to find, for each $i < j$, functions satisfying (i) and (iii) whose Hessians span $\mathcal{U}_{ij}$.

Lemma 6.10. *Suppose $i < j$ and ℓ is a loop with $hol_\ell(A)$ nontrivial. Set $L = hol_\ell(A) \in SU(3)$. Suppose further that $a, b \in \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$, and set $\xi_k = \int_{\gamma_k} a \in \mathfrak{h}^\perp$ and $\zeta_k = \int_{\gamma_k} b \in \mathfrak{h}^\perp$ for $k = 1, \ldots, m$. Then*

(i) Hess $tr\, hol_{\gamma_i \cdot \gamma_j}(A)(a,b) = tr((\xi_i + \xi_j)(\zeta_i + \zeta_j))$

(ii) Hess $tr\, hol_{\gamma_i \cdot \ell^{-1} \cdot \gamma_j \cdot \ell}(A)(a,b) = tr((\xi_i + L\xi_j L^{-1})(\zeta_i + L\zeta_j L^{-1}))$.

(iii) Hess $tr\, hol_{\ell \cdot \gamma_i \cdot \gamma_j}(A)(a,b) = tr(L(\xi_i + \xi_j)(\zeta_i + \zeta_j))$

Proof. By symmetry, it is enough to prove (i)–(iii) in the case $a = b$. For (i), this is just the statement that

$$\text{Hess}\, tr\, hol_{\gamma_i \cdot \gamma_j}(A)(a,a) = tr((\xi_i + \xi_j)^2)$$

for all $a \in \mathcal{H}^1_A(X; \mathfrak{h}^\perp)$, which follows directly from Corollary 2.7 (ii) as in the proof of Lemma 6.9, using the additional fact that $\int_{\gamma_i \cdot \gamma_j} a = \xi_i + \xi_j$.

To prove (ii), set $\beta = \gamma_i \cdot \ell^{-1} \cdot \gamma_j \cdot \ell$ and parameterize β by the interval $[0,4]$ so that the subintervals $[0,1], [1,2], [2,3]$ and $[3,4]$ parameterize $\gamma_i, \ell^{-1}, \gamma_j$, and ℓ, respectively. Define the function $\alpha : [0,4] \to \mathfrak{h}^\perp$ associated to the 1-form a using (18).

Now $\alpha|_{[1,2]}$ is exact since $hol_{\ell^{-1}}(A)$ is nontrivial. Similarly, $\alpha|_{[3,4]}$ is exact. Thus

$$\int_0^4 \alpha(t)dt = \int_0^1 \alpha(t)dt + \int_2^3 \alpha(t)dt = \xi_i + L\xi_j L^{-1}$$

by (19). Using Corollary 2.7 again and noting that $hol_\beta(A)$ is trivial, it follows that

$$\begin{aligned} \text{Hess}\, tr\, hol_\beta(A)(a,a) &= \int_0^4 \int_0^s tr(\alpha(s)\alpha(t) + \alpha(t)\alpha(s))dtds \\ &= \int_0^4 \int_0^4 tr\, \alpha(s)\alpha(t)dtds = tr((\xi_i + L\xi_j L^{-1})^2). \end{aligned}$$

To prove part (iii), set $\beta = \ell \cdot \gamma_i \cdot \gamma_j$ and parameterize β by the interval $[0,3]$ so that the subintervals $[0,1], [1,2]$ and $[2,3]$ parameterize ℓ, γ_i and γ_j, respectively. Define the function $\alpha : [0,3] \to \mathfrak{h}^\perp$ associated to a using (18). Use (19) and the fact that $\alpha|_{[0,1]}$ is exact to conclude that

$$\int_0^3 \alpha(t)dt = \int_1^3 \alpha(t)dt = L^{-1}(\xi_i + \xi_j)L.$$

Now Corollary 2.7 implies that

$$\begin{aligned} \text{Hess}\, tr\, hol_\beta(A)(a,a) &= \int_0^3 \int_0^3 tr(L\,\alpha(s)\alpha(t))dtds \\ &= tr\left(\int_0^3 L\,\alpha(s)ds \int_0^3 \alpha(t)dt\right) = tr((\xi_i + \xi_j)^2 L), \end{aligned}$$

and this completes the proof of (iii). □

If $a_i = 0 = a_j$, then $\xi_i = 0 = \xi_j$ and it follows from (i)–(iii) above that the Hessians at A of the real and imaginary parts of $tr\, hol_{\gamma_i\cdot\gamma_j}$, $tr\, hol_{\ell^{-1}\cdot\gamma_i\cdot\ell\cdot\gamma_j}$, and $tr\, hol_{\ell\cdot\gamma_i\cdot\gamma_j}$ lie in $\mathcal{U}_{ij}$. Consider the gauge invariant functions $\mathcal{A} \to \mathbb{R}$ defined by $f = \mathfrak{Re}\, tr\, hol_{\gamma_i\cdot\gamma_j}$, $f_\ell = \mathfrak{Re}\, tr\, hol_{\ell^{-1}\cdot\gamma_i\cdot\ell\cdot\gamma_j}$ and $g_\ell = \mathfrak{Im}\, tr\, hol_{\ell\cdot\gamma_i\cdot\gamma_j}$. Then conditions (i) and (iii) of Proposition 6.8 are satisfied for f, f_ℓ and g_ℓ. Condition (i) obviously holds, and condition (iii) follows from Corollary 2.7 just as in the case $i = j$ since the loops for f and f_ℓ (coming from parts (i) and (ii) of Lemma 6.10) have trivial holonomy and since g_ℓ is the *imaginary* part of trace of holonomy.

So, to complete the proof of 6.8, we just need to show that we can span $\mathcal{U}_{ij}$ with the Hessians of such functions. Restricting elements in $\mathcal{U}_{ij}$ to $V_i \times V_j$ we obtain 4×4 matrices as in (17). In contrast to the previous case when $i = j$, these matrices are not generally symmetric.

Suppose $a \in V_i$ and $b \in V_j$. Then $\xi_j = 0$ and $\zeta_i = 0$. Let $v = \varphi(\xi_i) \in \mathbb{C}^2$ and $w = \varphi(\zeta_j) \in \mathbb{C}^2$. If $\hat{L} = hol_\ell(\hat{A}) \in SU(2)$, (so $L = \hat{L} \oplus 1$), then Lemma 6.10 implies that

$$\begin{aligned}
\text{Hess}\, f(A)(a,b) &= tr(\xi_i\zeta_j) = -2\mathfrak{Re}\langle v, w\rangle, \\
\text{Hess}\, f_\ell(A)(a,b) &= tr(\xi_i\, L\, \zeta_j L^{-1}) = -2\mathfrak{Re}\langle v, \hat{L}(w)\rangle, \\
\text{Hess}\, g_\ell(A)(a,b) &= \mathfrak{Im}\, tr(L\, \xi_i\zeta_j) = -\mathfrak{Im}(\langle \hat{L}(v), w\rangle + \langle w, v\rangle).
\end{aligned}$$

Writing $\hat{L} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ where $\alpha = r + is$ and $\beta = t + iu$, then in terms of the real basis $\left\{ \binom{1}{0}, \binom{i}{0}, \binom{0}{1}, \binom{0}{i} \right\}$ for $\mathbb{C}^2$, the bilinear pairing $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by $(v, w) \mapsto \langle v, \hat{L}(w)\rangle$ has real part represented by the matrix

$$\Phi(\hat{L}) = \begin{bmatrix} r & -s & t & -u \\ s & r & u & t \\ -t & -u & r & s \\ u & -t & -s & r \end{bmatrix}. \tag{21}$$

Likewise, the bilinear pairing $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by $(v, w) \mapsto \langle \hat{L}(v), w \rangle + \langle w, v \rangle$ has imaginary part represented by the matrix

$$\Psi(\hat{L}) = \begin{bmatrix} s & 1-r & u & t \\ r-1 & s & -t & u \\ u & -t & -s & 1-r \\ t & u & r-1 & -s \end{bmatrix}. \tag{22}$$

Notice that the images of $SU(2)$ under Φ and Ψ span complementary 4-dimensional subspaces of the 8-dimensional space of matrices of the form (17).

Choose $x, y \in \pi_1(X)$ as before so that $\varrho(x)$ and $\varrho(y)$ do not commute. We first claim that the Hessians at A of f, f_x, f_y and f_{xy} are linearly independent. In fact, after restricting to $V_i \times V_j$, they span the 4-dimensional subspace of matrices (21). To show this, one only needs to show that the image of the set $\{I, \varrho(x), \varrho(y), \varrho(xy)\}$ under Φ is linearly independent. Again, this follows from the hypotheses on $\varrho(x)$ and $\varrho(y)$ easily after assuming (by conjugation) that $\varrho(x)$ is diagonal.

The complementary 4-dimensional subspace of $\mathcal{U}_{ij}$ given by (22) can be spanned using functions g_ℓ. The image of the set $\{I, \varrho(x), \varrho(y), \varrho(xy)\}$ under Ψ is linearly dependent because $\Psi(I) = 0$. However, a straightforward check shows that the image of $\{\varrho(x), \varrho(x^2), \varrho(y), \varrho(xy)\}$ under Ψ is linearly independent. Hence, it follows that the Hessians of g_x, g_{x^2}, g_y and g_{xy} are linearly independent. Since their span is complementary to that of the Hessians at A of f, f_x, f_y and f_{xy}, together they span $\mathcal{U}_{ij}$ and this completes the proof of Proposition 6.8. □

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Department of Mathematics, Ohio State University, Mansfield, OH, 44906

E-mail address: `boden@math.ohio-state.edu`

Department of Mathematics, Swarthmore College, Swarthmore, PA, 19081

E-mail address: `cherald1@swarthmore.edu`