Transversality For Equivariant Exact 1-Forms and Gauge Theory on 3-Manifolds

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Abstract

An equivariant jet transversality framework is developed for the study of critical sets of invariant functions on G manifolds. Techniques are developed to extend transversality results to the infinite dimensional Fredholm setting. As an application, the generic structure of the $SU(4)$ perturbed flat moduli space of an integral homology 3-sphere is described, as well as the generic structure of the parameterized moduli space for a path of perturbations. A similar analysis of the $U(3)$ moduli space for rational homology 3-spheres is also carried out.

Key words: Casson invariant, critical sets, symmetry, bifurcation
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1 Introduction

Let X be a smooth, closed, oriented 3-dimensional manifold, G be a compact Lie group, and $G \to P \to X$ be a principal G bundle over X. The purpose of this paper is to examine in detail transversality issues concerning the moduli space of flat connections on P. A fundamental tool for understanding the flat moduli space (or, more generally, the space of solutions to a Fredholm equation) is the Kuranishi technique. This technique will be discussed at some length in later sections, but in this introduction we give the following outline of the Kuranishi method.

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Let A be a flat connection on P. Any other connection differs from A by a differential 1-form with values in the adjoint bundle, $a \in \Omega^1(X; adP)$. To analyze the space of flat connections modulo gauge, at least locally, we can restrict our attention to the slice through A for the gauge group action, namely,

$$\{A + a \mid a \in \ker d_A^* \cap \Omega^1(X; adP)\}.$$

Furthermore, the Kuranishi model allows us to model the set of flat connections in the slice, those for which the curvature 2-form vanishes, as the zero set of a map from $\ker L$ to $\operatorname{coker} L$, where L is the linearization of the curvature map. (We are being slightly dishonest here, in that one must adjust this map to make the cokernel finite dimensional; see Proposition 61 for details.) The kernel and cokernel are identified with $H^1_A(X; adP)$ and $H^2_A(X; adP)$, the cohomology of X with coefficients in the bundle adP with its flat structure determined by A. These two cohomology groups are isomorphic, by Poincaré Duality.

One immediate consequence is that if $H^1_A(X; adP) = 0$, then the gauge orbit $[A]$ is a 0-dimensional component of the flat moduli space, nondegenerate in the sense that the orbit is cut out transversely by the equation "curvature equals zero." More generally, the dimension of $H^1_A(X; adP)$ gives an upper bound for the dimension of the moduli space near A.

There is considerably more information contained in the Kuranishi model. The flat moduli space near $[A]$ is identified with the zero set of a map $\Phi : H^1_A(X; adP) \to H^2_A(X; adP)$, modulo the action of $stab(A)$, the isotropy subgroup of A in the gauge group. This isotropy group is identified with the centralizer of the holonomy subgroup of A in G. The action of $stab(A)$ on the vector space $H^k_A(X; adP), k = 1, 2$ is linear, and the map Φ is equivariant. Furthermore, Φ can be identified with the differential of a certain invariant function $c_0 : H^1_A(X; adP) \to \mathbb{R}$.

If the flat moduli space fails to be a nondegenerate 0-dimensional manifold, the framework of "admissible holonomy perturbations" developed by Floer and Taubes provides a method to perturb the flatness equation without destroying its Fredholm property. In the $G = SU(2)$ setting, for X satisfying $H_1(X; \mathbb{Z}) = 0$, these perturbations were shown in [20] to have the property that a generic set of perturbations determine nondegenerate perturbed flat moduli spaces. Furthermore, deforming the perturbation along a generic path produces a cobordism between the perturbed flat moduli spaces corresponding to the endpoints.

For higher rank Lie groups G or more general 3-manifolds, the transversality problems are more complicated, due to the existence of various types of reducible flat connections (connections with different isotropy subgroups). The quotient space of connections modulo gauge has singularities at the orbits of the reducible connections. The main purpose of this paper is to develop the perturbation framework to address transversality questions related to these singularities.

In this paper we consider the following two cases:

(A) $G = U(3)$ or $SU(3)$ and X is a rational homology sphere ($\mathbb{Q}$HS), and
(B) $G = U(4)$ or $SU(4)$ and X is an integral homology sphere ($\mathbb{Z}$HS).

In both of these cases, we'll show that generic perturbations determine non-degenerate perturbed flat moduli spaces, and furthermore we will describe the structure of the parameterized moduli space for a generic path of perturbations. The latter result will provide the basis for Cerf-theoretic proofs that $SU(4)$ and $SU(3)$ Casson-type invariants are well-defined (perturbation independent) in these two cases, but we do not take up this task in this paper. The techniques developed here offer a promising avenue to prove similar results for more general groups G and for a broader class of 3-manifolds, but these generalizations require further calculations and will also not be discussed here.

Most of the work in this paper will be concerned with the Kuranishi model around a fixed flat connection. Ultimately, global results on moduli spaces will be obtained using compactness of the flat moduli space to obtain a cover by finitely many Kuranishi model neighborhoods. But, because so much of the work involves only the local structure, we will in large part abandon the usual terminology and symbols for the spaces, functions and maps involved in the 3-manifold gauge theory context for a simpler, more general approach.

Since we expect that many of the readers are already familiar with the terminology of connections, cohomology, curvature, the Chern-Simons function, etc., it may be useful in this introduction to translate between the usual terminology and the terminology used in the main body of this paper. This correspondence is made more precise in Section 6.

Name	Gauge Theory Symbol	Simpler Symbol
slice tangent space	$\mathcal{X}_A = \ker d_A^* \cap \Omega^1(X; adP)$	E
curvature map	$F : L^2_1 \mathcal{X}_A \to L^2 (*\mathcal{X}_A)$	$\zeta : E \to F$
first cohomology	$H^1_A(X; adP)$	U
second cohomology	$H^2_A(X; adP)$	W
isotropy subgroup	$stab(A)$	H
Chern-Simons function	$\mathrm{cs} : (A + L^2_1 \mathcal{X}_A) \to \mathbb{R}$	$c : E \to \mathbb{R}$

In summary, the local analysis in this paper is concerned with H invariant functions on an H vector space U. The groups H with which we'll be concerned are the stabilizers of the different types of reducible $U(n)$ connections, $n = 3, 4$.

To keep track of the possible reductions, we adopt the following notation. An n type connection is an irreducible connection of rank n. An $r_1 + r_2$ type connection is a gauge equivalent to $A_1 \oplus A_2$, where A_1 and A_2 are irreducible gauge inequivalent connections of rank r_1 and r_2, respectively. It has stabilizer isomorphic to $U(1) \times U(1)$ in $U(r_1 + r_2)$. Similarly, an $r_1 + r_2 + r_3$ connection may be decomposed into three inequivalent summands, and has stabilizer isomorphic to $U(1) \times U(1) \times U(1)$. If A is gauge equivalent to $A_1 \oplus A_1$, where A_1 is an irreducible $U(r_1)$ connection, then we say A has type r_1^2. The stabilizer of an r_1^2 connection in $U(2r_1)$ is isomorphic to $U(2)$. Similarly, an $r_1 + r_2^2$ type connection is a reducible connection gauge equivalent to $A_1 \oplus A_2 \oplus A_2$, where A_1 and A_2 are irreducible of rank r_1 and r_2, respectively. Such a connection has stabilizer in $U(r_1 + 2r_2)$ isomorphic to $U(1) \times U(2)$.

For concreteness, we now list the groups H and vector spaces U which arise in the cases (A) and (B) listed above. One can sometimes replace $stab(A)$ by a simpler group without altering the transversality issues, so in this list H may not correspond exactly to $stab(A)$. In the following table, $\underline{\mathbb{R}}^k$ and $\underline{\mathbb{R}}^l$ denote trivial representations of H (of arbitrary dimensions), $\mathbb{R}^3$ denotes the standard representation of $SO(3)$, and the complex vector spaces denote the obvious representations of the unitary groups indicated with the exception of the last one. Here $(\lambda, \lambda', \lambda'') \in U(1) \times U(1) \times U(1)$ acts on $(x, z, z', z'') \in \underline{\mathbb{R}}^k \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2} \oplus \mathbb{C}^{l_3}$ by

$$(\lambda, \lambda', \lambda'')(x, z, z', z'') = \left(x, \lambda'(\lambda'')^{-1}z, \lambda(\lambda'')^{-1}z', \lambda(\lambda')^{-1}z''\right).$$

Connection type	H	U
n	$\{1\}$	$\underline{\mathbb{R}}^k$
$r_1 + r_2$	$U(1)$	$\underline{\mathbb{R}}^k \oplus \mathbb{C}^l$
r_1^2	$SO(3)$	$\underline{\mathbb{R}}^k \oplus \left(\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l\right)$
$r_1 + 1^2$	$U(2)$	$\underline{\mathbb{R}}^k \oplus \left(\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l\right)$
$r_1 + r_2 + r_3$	$U(1) \times U(1) \times U(1)$	$\underline{\mathbb{R}}^k \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2} \oplus \mathbb{C}^{l_3}$

Section 5 contains representation-specific calculations needed for transversality arguments in each of the types of H vector space listed in this table. In Section 6, we show that for each flat connection A in either case (A) or (B), the $stab(A)$

invariant functions on $H^1_A(X; adP)$ correspond to invariant functions on one of these H vector spaces.

We now state the main transversality result in this paper. Fix X and G as in Case (A) or (B). Let $\mathcal{A}$ denote the space of G connections on the fixed principal G bundle P, and let $\mathcal{G}$ be the corresponding gauge group. These are to be completed using appropriate Sobolev norms, which will be made precise later on in this paper. Let $\mathrm{cs} : \mathcal{A} \to \mathbb{R}$ denote the Chern-Simons function.

Theorem 1 *Under the assumptions above, there exists a $\delta > 0$ and a residual subset of the ball $B_\delta(\mathcal{F})$ consisting of elements f for which the perturbed flat moduli space $\mathcal{M}_f(X)$ is a nondegenerate compact 0-manifold, (i.e.* $\mathrm{cs} + f$ *is Bott-Morse on $\mathcal{A}/\mathcal{G}$, with finitely many isolated critical orbits).*

For two such perturbations f_0 and f_1, generic paths in $B_\delta(\mathcal{F})$ from f_0 to f_1 have the property that the parameterized flat moduli space is a compact union of arcs, each lying in a fixed orbit type stratum. The individual arcs are compact except for open ends that limit to distinct points in a more reducible stratum of the parameterized moduli space. (There may well also be compact closed components, i.e. circles.)

For $\delta > 0$ sufficiently small, the orbit of the trivial connection remains an isolated point in the perturbed flat moduli space and there are no abelian perturbed flat connections.

Remark 2 *The conclusion of the theorem, expressed in other words, is that for generic paths of small perturbations, each stratum of the perturbed flat moduli space changes by a 1-dimensional cobordism. This cobordism is compact except for a specific type of bifurcation, which is topologically a "T"-intersection, wherein a less reducible perturbed flat orbit pops out of (or disappears into) an arc of more reducible perturbed flat orbits. This only happens for certain combinations of orbit types, indicated in Figure 1 for Case (A) and Figure 2 for Case (B).*

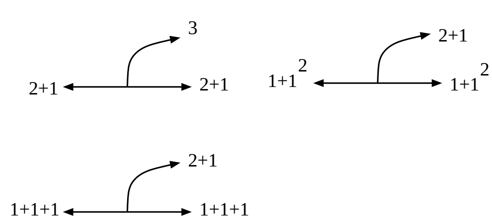

Fig. 1. These are the only "T"-intersection bifurcations for $U(3)$ or $SU(3)$ connections on a $\mathbb{Q}$HS, Case (A).

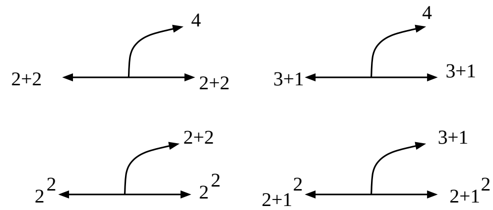

Fig. 2. These are the only "T"-intersection bifurcations for $U(4)$ or $SU(4)$ connections on a $\mathbb{Z}$HS, Case (B).

2 Jet Transversality and the Kuranishi Method

In this section we will establish a number of preliminary local results concerning transversality for Fredholm maps which are exact 1-forms. A standard technique in gauge theory known as the Kuranishi method (also called Liaponov-Schmidt reduction) allows one to model the zero set of a Fredholm map locally with a finite dimensional model. This section is concerned with perturbations of the Fredholm map and their effect on the Kuranishi model. Specifically, we begin with a Fredholm exact 1-form, and we modify it by adding the differential of another function to it. A fundamental question is the following. What condition on the space of perturbation functions guarantees that the corresponding perturbations of the finite dimensional Kuranishi model satisfy the standard jet transversality conditions (so that generic perturbations exhibit generic behavior)?

If the exact 1-forms are required to be equivariant (i.e. they are required to be differentials of invariant functions) under some group action, then the transversality questions become more delicate. The main technical results in this paper are jet transversality results about equivariant Fredholm exact 1-forms. This section lays out the nonequivariant (and hence simpler) version of these results, establishes notation, and provides a warm-up for the equivariant version.

2.1 Fredholm maps and the Kuranishi Method

We begin by recalling the Kuranishi method for studying the local structure of the zero set of a Fredholm map. Let E and F be Banach spaces, and let $\zeta : E \to F$ be a C^s Fredholm map with $s \geq 1$.

Consider a point $x \in \zeta^{-1}(0)$. For simplicity of notation, we will assume $x = 0$. Set $U = \ker d\zeta_0$ and choose a complementary subspace $V \subset E$. Similarly, set $V' = \text{Image}(d\zeta_0)$ and choose a complementary subspace $W \subset F$. Let $\Pi_{V'} : F \to V'$ and $\Pi_W : F \to W$ denote the projections associated to the direct sum decomposition $F = V' \oplus W$.

Proposition 3 *The two projections* $\Pi_{V'} : F \to V'$ *and* $\Pi_W : F \to W$ *are bounded.*

PROOF. Consider the map $V' \oplus W \to F$ given by $(v, w) \mapsto v + w$. It is bounded by the triangle inequality. Because $V' + W = F$ and $V' \cap W = 0$, this map is also 1-1 and onto. Banach's theorem states that if a bounded linear map between Banach spaces is invertible, then its inverse is also bounded, so $\Pi_{V'} \oplus \Pi_W : F \to V' \oplus W$ is bounded. $\square$

Unless $d\zeta_0$ is surjective (i.e. $W = 0$), we cannot apply the Implicit Function Theorem to conclude that $Z = \zeta^{-1}(0)$ is a manifold near 0. The Kuranishi method is to apply the Implicit Function Theorem to gain as much information as we can, in spite of this failure. Namely, we will capitalize on the fact that the differential of $\Pi_{V'} \circ \zeta$ at 0, restricted to the V summand, is an isomorphism

$$D_2 \left(\Pi_{V'} \circ \zeta\right)_0 = \Pi_{V'} \circ D_2\zeta_0 : V \to V',$$

by our choice of decompositions of E and F.

Let $P : E = U \oplus V \to V'$ be the map defined by

$$P(u, v) = \Pi_{V'} \circ \zeta(u, v).$$

(If we were studying Z near a point $z \neq 0$, we would set $P(u, v) = \Pi_{V'} \circ \zeta(z + (u, v))$.) Since $D_2 P_0$ is surjective, $P^{-1}(0)$ is a C^s manifold near $(0, 0)$ which can be parameterized as the graph of a smooth map $\psi : B_U \subset U \to V$. That is, there is a product neighborhood $B_U \times B_V \subset U \oplus V$ of $(0, 0)$ such that

$$Z \cap (B_U \times B_V) = \{(u, \psi(u)) \mid u \in B_U\}.$$

A finite dimensional model for Z near 0 is now obtained as follows. Let $\Phi : B_U \to W$ be the map

$$\Phi(u) = \Pi_W \circ \zeta(u, \psi(u)).$$

Since $\zeta(x)$ vanishes if and only if both projections $\Pi_{V'} \circ \zeta(x)$ and $\Pi_W \circ \zeta(x)$ vanish, the zeros of ζ in $B_U \times B_V$ are exactly the points $(u, \psi(u))$ for which $\Phi(u) = 0$. In other words, $Z \cap (B_U \times B_V)$ is identified, under the graphing map $\Psi : u \mapsto (u, \psi(u))$, with the zero set of $\Phi : B_U \subset U \to W$, a C^s map between finite dimensional vector spaces.

In the next subsection, we consider the situation where $F \subset E^*$ and the map $\zeta : E \to F$ is the differential of a C^{s+1} function $c : E \to \mathbb{R}$. That is, $\zeta(x)$ is the element of E^* given by $\zeta(x)(\vec{x}) = \frac{d}{dt} c(x + t\vec{x}) \mid_{t=0}$.

2.2 Fredholm exact 1-forms

In this subsection, we specialize the discussion of the last subsection to the case where the Fredholm map ζ is an exact 1-form on E. (Everything we say here applies to a finite dimensional manifold, although the nonlinear abundance questions are not entirely well-motivated in this context. This is a recommended warm-up case for the reader to consider.)

Let $c : E \to \mathbb{R}$ be a C^{s+1} function, and let $dc : E \to E^*$ denote its differential. Assume that dc takes values in $F \subset E^*$ and that the map $dc : E \to F$ is Fredholm. Then the map $dc : E \to F$ may be taken to be ζ as in Section 2.1, giving rise to local Kuranishi descriptions of the critical set of c (the zero set of dc). In this subsection, we will explore the Kuranishi method in this context of exact 1-forms more fully.

Let $x \in \mathrm{crit}(c) = (dc)^{-1}(0)$. For simplicity of notation, we assume $x = 0 \in E$. Let $D(dc)_0$ denote the linearization of the exact 1-form $dc : E \to F$ at $x = 0$. We now make several other assumptions about c.

Hypothesis A. We assume that F is a Banach space contained in E^*, that dc takes values in F, and that $dc : E \to F$ is a C^s Fredholm map. Set $U = \ker D(dc)_0 \subset E$ and $V' = \mathrm{Image}\,(D(dc)_0) \subset F$. We assume that there exist complementary spaces V and W with the following properties:

- $E = U \oplus V$ and $F = W \oplus V'$, and
- $W = U^*$ and $V' \subset V^*$, i.e. for each $w \in W$ and $v \in V$, $w(v) = 0$.

Remark 4 *In the gauge theory applications we have in mind, E is a subspace of the space of connections on a principal bundle over a closed 3-manifold X, completed with respect to a Sobolev norm L^2_1. Fixing a base connection A, we can identify E with $L^2_1\Omega^1(X; adP)$ (more precisely, with the slice tangent space for the gauge group action through A, $L^2_1\Omega^1(X; adP) \cap \ker d^*_A$). The dual space is $L^2_{-1}\Omega^2(X; adP)$. The pairing is given by*

$$(\omega_1, \omega_2) \mapsto \int_X -tr(\omega_1 \wedge \omega_2),$$

where $tr : Lie(G) \times Lie(G) \to \mathbb{R}$ is a negative definite, adjoint invariant symmetric bilinear form on the Lie algebra. (If $G \subset U(n)$, this can be taken to be the usual trace of the product.)

The function $c : E \to \mathbb{R}$ to be studied is the Chern-Simons function, which has the property that $dc_A = -\frac{1}{4\pi^2} F(A) \in L^2\Omega^2(X; adP)$, where $F(A)$ the curvature 2-form of A.

At a flat connection A, the linearization of the curvature map is the twisted de Rham operator $d_A : L^2_1\Omega^1(X; adP) \to L^2\Omega^2(X; adP)$, which is part of an elliptic complex. With suitable restriction of the domain and modification of the range (see Lemma 60 for details), we therefore obtain a Fredholm map when L^2 is used on the range. This explains our motivation for the hypothesis about $F \subset E^*$. We have a Fredholm map from L^2_1 to $L^2 \subset L^2_{-1} = (L^2_1)^*$.

In addition, after the domain and range modifications, after fixing a Riemannian metric on X, the kernel and cokernel of the map d_A above are the cohomology groups $U = H^1_A(X; adP)$ and $W = H^2_A(X; adP)$ with coefficients in the flat bundle determined by A, specifically the spaces of harmonic representatives for the cohomology. The harmonic representatives are smooth, and hence these spaces are independent of the Sobolev norms. The metric on X gives rise to an L^2 inner product on forms, and we can choose V and V' to be orthogonal complements to the harmonic representatives. Then the second part of Hypothesis A holds also.

Set $P : U \oplus V \to V'$ equal to

$$P(u, v) = \Pi_{V'}\left(dc_{(u,v)}\right).$$

Note that $P(u, v) = 0$ if and only if $dc_{(u,v)}$ annihilates V. Near the critical point $x = 0$, $P^{-1}(0)$ is parameterized by $\{(u, \psi(u)) \mid u \in B_U\}$ for a map $\psi : B_U \to V$. Again there is a map $\Phi : B_U \to W$ given by

$$\Phi(u) = \Pi_W\left(dc_{(u,\psi(u))}\right),$$

with the property that the critical set of c near zero is the image of $\Phi^{-1}(0)$ under the map $\Psi : u \mapsto (u, \psi(u))$.

The following proposition identifies the Kuranishi map Φ as an exact 1-form on a finite dimensional space.

Proposition 5 *Under Hypothesis A, Φ is the differential of the C^s function $\tilde{c} : B_U \to \mathbb{R}$ defined by*

$$\tilde{c}(u) = c(u, \psi(u)).$$

PROOF. Let D_1c and D_2c denote the differentials of c with respect to the summands in $E = U \oplus V$, i.e. $\Pi_W(dc)$ and $\Pi_{V'}(dc)$. Then ψ is defined implictly by the equation $D_2c_{(u,\psi(u))} = 0$. Therefore, for $\vec{u} \in T_uU$, we have

$$\begin{aligned} d\tilde{c}_u(\vec{u}) &= D_1c_{(u,\psi(u))}(\vec{u}) + D_2c_{(u,\psi(u))} \circ D\psi_u(\vec{u}) \\ &= \Pi_W(dc_{(u,\psi(u))})(\vec{u}). \end{aligned}$$

□

We now consider the effect on the zero set, and on the local Kuranishi description of this set, of perturbing a Fredholm map. To begin with, we return to the general setting of a C^s Fredholm map $\zeta : E \to F$. Let $\mathcal{E}$ be a Banach space, with a C^s map $\mathcal{S} : E \times \mathcal{E} \to F$. We will view $\mathcal{E}$ as a space of perturbations of ζ, associating to each $e \in \mathcal{E}$ the map

$$\zeta_e = \zeta + \mathcal{S}(e) : E \to F,$$

where $\mathcal{S}(e)(x) = \mathcal{S}(x, e)$.

Proposition 6 *For any compact set $K \subset E$, there is a neighborhood $N \subset E$ of K and a $\delta > 0$ such that for each $e \in \mathcal{E}$ with $\|e\| < \delta$,*

$$\zeta_e|_N : N \to F$$

is Fredholm.

PROOF. First fix a point $x \in K$. The linearization $d\zeta_x : T_xE \to T_{\zeta(x)}F$ is a Fredholm linear map. The space of Fredholm linear maps is an open subset of the space of bounded linear maps $L(T_xE, T_{\zeta(x)}F)$. It follows that for sufficiently small $e \in \mathcal{E}$, $d(\zeta_e)_x : T_xE \to T_{\zeta(x)}F$ is Fredholm.

Viewed as an element of $L(E, F)$, $d(\zeta_e)_x$ varies continuously in x and in e, so it follows that there exists $\delta_1 > 0$ and $\delta_2 > 0$ with the property that $\|e\| < \delta_1$ and $\|x - x'\| < \delta_2$ insure that $d(\zeta_e)_{x'}$ is Fredholm.

Choosing such δ_1 and δ_2 for each $x \in K$, we can take a finite subcover of the δ_2 balls for K, setting N to be the union. Let δ be the minimimum over the finite set of corresponding δ_1 values. The result follows. $\square$

We will, for the time being, be interested in local results, so we will fix a point in $\zeta^{-1}(0)$, which we again assume to be $0 \in E$ for simplicity. Applying Proposition 6 to $K = \{0\}$, we get neighborhoods $B_{\mathcal{E}}$ and B_E such that $\zeta_e : B_E \to F$ is Fredholm for all $e \in B_{\mathcal{E}}$.

With the same decompositions of E and F as before, extend the map P to a map

$$P : B_U \times B_V \times B_{\mathcal{E}} \to V',$$

$$P(u, v, e) = \Pi_{V'} \circ \zeta_e(u, v). \tag{1}$$

Again, the V differential is surjective, so we obtain (after possibly shrinking the neighborhoods) a C^s map $\psi : B_U \times B_{\mathcal{E}} \to V$ such that

$$\begin{aligned} &P^{-1}(0) \cap (B_U \times B_V \times B_{\mathcal{E}}) \\ &= \{(u, \psi(u,e), e) \mid (u,e) \in B_U \times B_{\mathcal{E}}\}. \end{aligned}$$

Now the universal zero set $\{(u,v,e) \in B_U \times B_V \times B_{\mathcal{E}} \mid \zeta_e(u,v) = 0\}$ is identified with the zero set of the C^s map $\Phi : B_U \times B_{\mathcal{E}} \to W$ defined by

$$\Phi(u,e) = \Pi_W \circ \zeta_e(u, \psi(u,e)). \tag{2}$$

Again, we will let Ψ denote the graphing map $\Psi : B_U \times B_{\mathcal{E}} \to U \times V \times \mathcal{E}$,

$$\Psi(u,e) = (u, \psi(u,e), e).$$

We now assume, once again, that the Fredholm map ζ is an exact 1-form on E, $\zeta = dc : E \to F \subset E^*$. We consider perturbing by adding other exact 1-forms. We make the following additional assumptions.

Hypothesis B. We assume that

- $\mathcal{F}$ is a Banach space contained in $C^{s+1}(E)$.
- For all $f \in \mathcal{F}$, the differential df takes values in F, i.e. $df : E \to F \subset E^*$.
- The map $(x, f) \in E \times \mathcal{F} \mapsto df_x \in F$ is C^s.

Note that the general discussion above concerning perturbations of Fredholm maps applies in this context, with $\mathcal{E} = \mathcal{F}$ and $\mathcal{S}(f) = df : E \to F$. Thus if we define a map P as in equation (1) (with $\mathcal{E}$ replaced by $\mathcal{F}$), we obtain maps ψ, Φ, and Ψ as in equation (2). Proposition 5 generalizes immediately to the following statement.

Proposition 7 *The parameterized Kuranishi map $\Phi : B_U \times B_{\mathcal{F}} \to W$ is given by $\Phi(u,f) = d\tilde{f}(u)$, where $\tilde{f} : B_U \to \mathbb{R}$ is $\tilde{f}$ is the C^s function defined (in terms of the implicitly defined map ψ) by*

$$\tilde{f}(u) = (c + f)(u, \psi(u,f)).$$

The content of Proposition 7 is that via the Kuranishi method, the study of critical sets cut out by Fredholm exact 1-forms reduces to the study of the critical sets of functions on a finite dimensional space. We next focus on transversality issues concerning critical sets in the latter setting. These issues are conveniently discussed in terms of jets, so we pause to establish some useful notation and recall the jet transversality theorem.

2.4 Jet transversality

In this subsection, let U be a finite dimensional normed vector space. Let $C^k(U)$ denote the set of (real valued) C^k functions on U, and let $C^k_{(b)}(U)$ denote the set of functions with all derivatives of order less than or equal to k bounded. This can be made into a Banach space by defining a norm

$$\|f\| = \sum_{j=0}^{k} \sup_{u \in U} |d^j f_u|,$$

where the terms in the sum are the operator norms of $d^j f_u : U^{\otimes j} \to \mathbb{R}$.

Definition 8 *For any integer $k \geq 0$, let $J^k(U)$ denote the set of k-jets of maps from U to $\mathbb{R}$. Let $\alpha : J^k(U) \to U$ be the source map, and set $J^k(U)_u = \alpha^{-1}(u)$. Assume $r > 0$ and define $j^k : C^{r+k}(U) \times U \to J^k(U)$ by*

$$j^k(f, u) = j^k_u(f) = [u, f] \in J^k(U)_u.$$

We will sometimes view j^k as a map $j^k : C^{r+k}(U) \to C^r(U, J^k(U))$. (See [15], for example.)

Notice that $J^k(U)$ is a vector bundle over U and that, if f is C^k, $j^k(f)$ defines a section of it. To illustrate the usefulness of jets, we recall the Jet Transversality Theorem and a number of applications. See Theorems 3.2.7 and 3.2.8 in [15] for a more general discussion of these issues. We will adapt the theorem in [15] to the present setting of a (nonlinear) C^r map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$. The reader should keep in mind the implicitly defined $\mathcal{T}(f) = \tilde{f}$ from Proposition 7.

Although for simplicity we will state these results for globally defined functions on U, they generalize easily to open subsets of U (at the expense of having to differentiate between the subset and its tangent space). Also, note that any C^s function on an open ball $B_\epsilon(U)$ is in $C^s_{(b)}(B_{\epsilon'}(U))$ for each $\epsilon' < \epsilon$, so the bounded derivatives condition is not a serious problem for local arguments.

Recall that a subset of a topological space is *residual* if it is a countable intersection of open dense sets.

Theorem 9 *Let $A \subset J^k(U)$ be a C^∞ submanifold, and fix $r > \max\{0, \dim U - \operatorname{codim} A\}$. Suppose that $\mathcal{F}$ is a Banach space and that $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$ is a C^r map. Let $Q : \mathcal{F} \times U \to J^k(U)$ be the C^r map defined by $Q(f, u) = j^k_u(\mathcal{T}(f))$.*

If Q is transverse to A, then

$$\{f \in \mathcal{F} \mid j^k(\mathcal{T}(f)) : U \to J^k(U) \text{ is transverse to } A\}$$

is residual in $\mathcal{F}$ and hence dense. For f in this residual subset,

$$\{u \in U \mid j^k_u(\mathcal{T}(f)) \in A\}$$

is a submanifold of U of codimension equal to $\operatorname{codim} A$.

Remark 10 *The hypothesis on r, which comes from the differentiability requirement for the Sard-Smale Theorem, is not a big problem for us. All of the submanifolds A with which we will be concerned are contained in $\{[u, f] \in J^k(U) \mid df_u = 0\}$, so $\operatorname{codim} A \geq \dim U$. Therefore it will be sufficient to take $r > 0$.*

PROOF. Set $W = Q^{-1}(A) \subset \mathcal{F} \times U$. By the Implicit Function Theorem, W is a C^r submanifold, of codimension equal to the codimension of A in $J^k(U)$. The projection $\pi : \mathcal{F} \times U \to \mathcal{F}$ is Fredholm, of index $\dim U$. Thus the restriction $\pi|_W : W \to \mathcal{F}$ is a Fredholm map of index $\dim U - \operatorname{codim} A$. By the Sard-Smale Theorem, the set of regular values is residual in $\mathcal{F}$, and one can check that f is a regular value of $\pi|_W$ if and only if $j^k(\mathcal{T}(f))$ is transverse to A. The preimage of a regular value f is $\{f\} \times \{u \in U \mid Q(f, u) \in A\}$, a submanifold of $\{f\} \times U$. $\square$

The following corollary illustrates the utility of this theorem in the study of critical sets. (This first corollary could be stated more efficiently in terms of the cotangent bundle instead of jet spaces, but we formulate it this way as a warm-up for the other applications that follow.)

Corollary 11 *Let $\mathcal{F}$ be a Banach space and let $k \geq 1$. Suppose that $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$ is a C^r map with the property that the map $Q : \mathcal{F} \times U \to J^k(U)$, defined by $Q(f, u) = j^k_u(f)$, is transverse to the submanifold*

$$A = \{[u, f] \in J^k(U) \mid df_u = 0\}.$$

Then

$$\{f \in \mathcal{F} \mid \mathcal{T}(f) \text{ is Morse}\}$$

is residual in $\mathcal{F}$ and hence dense.

PROOF. The codimension of A in $J^k(U)$ is $\dim U$. By Theorem 9, for f in a residual subset of $\mathcal{F}$,

$$\operatorname{crit}(\mathcal{T}(f)) = \{u \in U \mid j^1_u(\mathcal{T}(f)) \in A\}$$

is a 0-dimensional submanifold of U. Implicit in the final statement in the theorem is that the condition that $j^k_u(\mathcal{T}(f)) \in A$ cuts out its solution set in U transversely. This means that $d(\mathcal{T}(f)) : U \to T^*U$ has transverse zeros, so $\mathcal{T}(f)$ is Morse. $\square$

There is a standard method for extending these transversality results to families of functions in $\mathcal{F}$. We will be interested only in one dimensional families in this paper. Suppose that f_0 and f_1 are two elements of $\mathcal{F}$ such that $\mathcal{T}(f_0)$ and $\mathcal{T}(f_1)$ are Morse functions. Fix $r > \max\{0, \dim U - \operatorname{codim} A + 1\}$. Let $\widetilde{C}^{r+k}([0,1], \mathcal{F})$ denote the space of C^{r+k} paths from f_0 to f_1. This is an affine space modeled on the Banach space $\Big\{\{f_t\} \in C^{r+k}([0,1], \mathcal{F}) \mid f(0) = f(1) = 0\Big\}$. In all the cases where we apply this theorem, $r = 2$ will suffice.

Theorem 12 *With U, $A \subset J^k(U)$, $\mathcal{F}$, $\mathcal{T}$ and Q as above, the set*

$$\Big\{\{f_t\}_{0 \le t \le 1} \mid j^k\mathcal{T}(f_t) : U \times [0,1] \to J^k(U) \text{ is transverse to } A\Big\}$$

is residual in $\widetilde{C}^{r+k}_{(b)}([0,1], \mathcal{F})$. For any path in this residual set, the set

$$\{(u,t) \in U \times [0,1] \mid j^k_u(f_t) \in A\}$$

is a submanifold of $U \times [0,1]$ of codimension equal to $\operatorname{codim} A$.

PROOF. If the map Q is transverse to A, then clearly the extended map

$$\widetilde{Q} : \widetilde{C}^{r+k}_{(b)}([0,1], \mathcal{F}) \times U \times [0,1] \to J^k(U),$$

given by $\widetilde{Q}(\{f_t\}, u, t_0) = j^k_u(f_{t_0})$, is transverse to A. The path space is a Banach manifold, and we can make exactly the same argument as in Theorem 9. $\square$

Corollary 13 *Assume the transversality hypothesis of Theorem 12 holds for*

$$A = \{[u,f] \in J^k(U) \mid df_u = 0\}$$

for some $k \ge 1$. Then for a residual set of paths from f_0 to f_1, the parameterized critical set $\{(u,t) \in U \times [0,1] \mid d(f_t)_u = 0\}$ is a (possibly non-compact) 1-dimensional manifold with boundary equal to

$$(\{0\} \times \operatorname{crit}(f_0)) \cup (\{1\} \times \operatorname{crit}(f_1)).$$

PROOF. This is an immediate consequence of Theorem 12. $\square$

Remark 14 *This corollary does not preclude functions f_{t_0} in the path with non-isolated critical points. There may in fact be a whole 1-manifold of critical points for certain values of t. One can see this already with $U = \mathbb{R}$ and $\mathcal{F} = \{$linear functions$\}$. Consider, for example, the functions $f_0(x) = x$ and $f_1(x) = -x$, with the path $f_t(x) = (1-2t)x$ connecting them.*

Corollary 15 *Assume that, for some $k \geq 2$, the transversality hypothesis of Theorem 12 holds for*

$$A = \{[u, f] \in J^k(U) \mid df_u = 0\}$$

and for each of the subsets

$$A_l = \{[u, f] \in J^k(U) \mid df_u = 0 \text{ and } \dim\ker d^2 f_u = l\},$$

for $l = 1, \ldots, \dim U$. Then for a residual set of paths from f_0 to f_1, the points (u_0, t_0) in the 1-dimensional parameterized critical set

$$\{(u, t) \in U \times [0, 1] \mid d(f_t)_u = 0\}$$

where $\dim\ker d^2(f_t)_u = 1$ are isolated, and there are no points where this dimension is larger than one.

PROOF. This follows from the fact that A_1 has codimension one in A and, for $l > 1$, A_l has codimension > 1 in A. We leave this computation to the reader. We make a similar argument in the more complicated equivariant setting in Proposition 39. $\square$

Corollary 16 *Assume the transverality hypothesis of Theorem 12 holds for*

$$A, A_1, \ldots, A_{\dim U}, A' \subset J^k(U), \quad k \geq 3,$$

where $A, A_1, \ldots, A_{\dim U}$ are as defined above and

$$A' = \{[u, f] \in J^k(U) \mid df_u = 0, \quad \dim\ker d^2 f_u = 1, \text{ and } d^3 f_u|_{\ker d^2 f_u} = 0\}.$$

Then for a residual set of paths from f_0 to f_1, the 1-dimensional parameterized critical set has the following properties at each (u_0, t_0) in it where $d^2(f_{t_0})_{u_0}$ has nontrivial (1-dimensional) kernel:

(a) f_{t_0} *has the property that* $d^3(f_{t_0})|_{\ker d^2(f_{t_0})_{u_0}}$ *is nontrivial.*
(b) $\frac{\partial}{\partial t}(df_t)_{u_0}|_{t_0} \neq 0$.

PROOF. The proof of the first claim is another application of the Parameterized Jet Transversality Theorem, relying on the fact that $A' \subset A_1$ is a submanifold of codimension one.

The second claim follows from the fact that the parameterized critical set is a 1-manifold cut out transversely in $U \times [0,1]$. At (u_0, t_0), the tangent space must be $\ker d^2(f_{t_0})_{u_0} \oplus \{0\}$. This implies the second claim. $\square$

Corollary 17 *Under the hypothesis of the Corollary 16, the bifurcations in the critical set, for a generic path connecting Morse functions in $\mathcal{F}$, are "standard Morse births and deaths" which are modeled on the bifurcation in the critical set of $f_t : \mathbb{R} \to \mathbb{R}$, $f_t(x) = -tx + \frac{x^3}{3} + R(t,x)$ where $\lim_{x \to 0} \frac{R(t,x)}{x^3} = 0$.*

PROOF. The reduction to 1-dimension is a standard Kuranishi or Liaponov-Schmidt reduction. The two conditions in Corollary 16 insure that the reduced model has the same property, except now there is only one "space variable" and $\ker d^2 f_{t_0}$ is the whole line. The expansion of $f_t(x)$ now follows (after rescaling t and x to adjust the coefficients). $\square$

The idea behind jets is that they provide an intrinsic formulation of the Taylor expansion, independent of any coordinate system. Taylor expansions, on the other hand, behave in a rather complicated way with respect to composition with diffeomorphisms or other differentiable maps.

The following identification of the space of jets will be helpful. For any finite dimensional vector space U, let $S^k(U) \subset (U^*)^{\otimes k}$ denote the space of symmetric k tensors on U. (It will be convenient to set $S^0(U) = \mathbb{R}$.) We will use the notation $S^{0+\cdots+k}(U)$ to denote the direct sum $\bigoplus_{i=0}^{k} S^i(U)$. If U is a subset of a vector space, then one must make the straightforward modification of replacing $S^l(U)$ with $S^l(T_u U)$. Let $d^k(\cdot)_u : C^k(U) \to S^k(U)$ denote the kth derivative at u, sending f to $d^k f_u$ (define $d^0 f_u = f(u)$).

Using linear coordinates on U, we obtain a natural bijective correspondence between $J^k(U)_u$ and $S^{0+\cdots+k}(U)$, given by

$$[u, f] \mapsto (d^0 f_u, d^1 f_u, \ldots, d^k f_u). \tag{3}$$

In this notation, $d^0 f_u$ is simply the value $f(u)$. In this way, $J^k(U)$ becomes a (trivial) vector bundle over U with fiber $S^{0+\cdots+k}(U)$. (It is also tempting to view the fiber as the space of polynomials on U of degree less than or equal to k, associating $[u, f]$ to the Taylor expansion of f at u. This is a little more complicated, since the Taylor polynomial at u_0 is a polynomial in $u - u_0$. The former identification avoids these complications and works much better for our purposes.)

In all jet transversality applications that come up in this paper, we will be concerned with submanifolds A which are transverse to the fiber of $\alpha : J^k(U) \to$

U. Thus it will be helpful to have the following sufficient condition to show Q is transverse to A (which may well be stronger than necessary, but is independent of a particular choice of A).

Proposition 18 *Suppose, in Theorem 9, that $A \subset J^k(U)$ is transverse to the fibers of $\alpha : J^k(U) \to U$. Then to verify the transversality hypothesis that Q is transverse to A, it is sufficient to show that the first differential $D_1Q_{(f,u)} : T_f\mathcal{F} \to T_{j^k_u(\mathcal{T}(f))}J^kU$ maps onto the tangent space of the fiber $J^k(U)_u$.*

Equivalently, if we set $\mathcal{F}^{(-1)}_{(f,u)} = T_f\mathcal{F}$ and

$$\mathcal{F}^{(l)}_{(f,u)} = \{\delta f \in T_f\mathcal{F} \mid j^l_u\left(D\mathcal{T}_f(\delta f)\right) = 0\}, \quad 0 \leq l \leq k,$$

then it is sufficient to check that the maps

$$d^l_u \circ D\mathcal{T}_f : \delta f \in \mathcal{F}^{(l-1)}_{(f,u)} \mapsto d^l(D\mathcal{T}_f(\delta f))_u \in S^l(T_uU)$$

are surjective for $l = 0, \ldots, k$.

PROOF. Note that the image of $Q|_{\mathcal{F}\times\{u\}}$ lies in the fiber $J^k(U)_u$, so the image of the differential $D_1Q_{(f,u)} : T_f\mathcal{F} \to T_{j^k_u(\mathcal{T}(f))}J^k(U)$ lies in the tangent space of the fiber. Thus the first claim is obvious.

The second claim is equally straightforward. The key observation is that j^k_u is linear, so $D(j^k_u \circ \mathcal{T})_f(\delta f) = j^k_u\left(D\mathcal{T}_f(\delta f)\right)$. Any $\delta f \in \mathcal{F}^{(l-1)}_{(f,u)}$ contributes nothing under this map to the $S^j(U)$, $j < l$, summands in $S^{0+\cdots+k}(U)$ under the map (3). $\square$

For a linear map $L : U \to V$, there is a pullback map $L^* : S^k(V) \to S^k(U)$ given by $L^*\tau(u_1, \ldots, u_k) = \tau(Lu_1, \ldots, Lu_k)$.

Remark 19 *The advantage of filtering $T_f\mathcal{F}$ as we have in Proposition 18 has to do with simplifying chain rule computations. For example, if $h \in C^{r+k}(U)$ vanishes to order $l-1$ at u and $\phi : U \to U$ is a diffeomorphism sending u to u, then $h \circ \phi$ also vanishes to order $l-1$ at u, and furthermore the l-th derivative of h behaves tensorially:*

$$d^l(h \circ \phi)_u = (d\phi_u)^*\left(d^l h_u\right).$$

Here, $(d\phi_u)^*$ is the pullback of the linear map $d\phi_u : T_uU \to T_uU$.

With Theorem 9 and Proposition 18 as motivation, we make the following definition.

Definition 20 *Given a Banach space $\mathcal{F}$ and a C^r map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$, we will say $(\mathcal{T}, \mathcal{F})$ is* kth order abundant *at $(u, f) \in U \times \mathcal{F}$ if the map $j^k_u \circ \mathcal{T} : \mathcal{F} \to J^k(U)_u$ sending $(f, u) \mapsto j^k_u(\mathcal{T}(f))$ is a submersion at (u, f). This is equivalent to the surjectivity of the maps $d^l_u \circ D\mathcal{T}_f : \mathcal{F}^{(l-1)}_{(f,u)} \to S^l(U)$ for $l = 0, \ldots, k$ in the second part of Proposition 18.*

The following proposition shows that although Taylor expansions depend on a coordinate system, the property of kth order abundance does not.

Proposition 21 *If $\phi : U \to U$ is a $C^{r+k}_{(b)}$ diffeomorphism fixing u, then any map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$ is kth order abundant at (u, f) if and only if $\phi^*\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$, $\phi^*\mathcal{T}(f) = \mathcal{T}(f)\circ\phi$, is kth order abundant at (u, f). If $\phi(u) \neq u$, then abundance of $\mathcal{T}$ at (u, f) is equivalent to abundance of $\phi^*\mathcal{T}$ at $(\phi^{-1}(u), f)$.*

PROOF. If $(\mathcal{T}, \mathcal{F})$ is abundant at (u, f) then, for each $0 \leq l \leq k$, we can choose $\{\delta f^l_i\}_{1 \leq i \leq m_l} \in T_f\mathcal{F}$ such that

- for $j < l$, $d^j \left(D\mathcal{T}_f(\delta f^l_i)\right)_u = 0$, and
- $\left\{d^l \left(D\mathcal{T}_f(\delta f^l_i)\right)_u\right\}_{0 \leq i \leq m_l}$ span $S^l(U)$.

The right composition map, $\phi^* : C^{r+k}_{(b)}(U) \to C^{r+k}_{(b)}(U)$, $h \mapsto h \circ \phi$, is linear, so $D(\phi^*\mathcal{T})_f(\delta f) = \phi^*(D\mathcal{T}_f(\delta f))$. If $D\mathcal{T}_f(\delta f)$ vanishes to order $l - 1$ at u, then

$$d^j \left(\phi^*(D\mathcal{T}_f(\delta f))\right)_u = \begin{cases} 0 & j < l \\ (d\phi_u)^* \quad d^l \left(D\mathcal{T}_f(\delta f)\right)_u & j = l, \end{cases}$$

and $(d\phi_u)^* : S^l(T_uU) \to S^l(T_uU)$ is an isomorphism. Applying this to the tangent vectors δf^l_i above easily proves the proposition. $\square$

2.5 Abundance for perturbations of the Kuranishi model

We now return to the setting where $\mathcal{F} \subset C^{r+k+1}(U \oplus V)$ and $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b)}(U)$ is defined implicitly in terms of ψ by

$$\begin{cases} \mathcal{T}(f)(u) = (c + f)(u, \psi(u, f)), \\ \quad D_2(c + f)(u, \psi(u, f)) = 0 \end{cases} \tag{4}$$

(on a neighborhood $B_U \times B_{\mathcal{F}}$ of $(0, 0)$, but we will not usually refer to the neighborhood). This subsection is concerned with how jets of the functions f

are related to jets of the corresponding functions $\mathcal{T}(f) = \tilde{f}$. More precisely, we'll consider the following two questions:

- For a map $\mathcal{T} : \mathcal{F} \subset C^{r+k+1}(U \oplus V) \to C^{r+k}_{(b)}(B_U)$ of the form in equations (4), what minimal condition on $\mathcal{F}$ guarantees that $(\mathcal{T}, \mathcal{F})$ is kth order abundant on $B_U \times B_{\mathcal{F}}$?
- What condition on $(\mathcal{T}, \mathcal{F})$ guarantees abundance on a possibly smaller neighborhood of $(0,0) \in U \times \mathcal{F}$?

For $\mathcal{T}$ of this special form, the following result will be an important calculational tool.

Proposition 22 *For $\mathcal{T}$ and ψ defined as in equations (4),*

$$D\mathcal{T}_f(\delta f)(u) = \delta f(u, \psi(u, f)).$$

PROOF. In the notation on the left side of the equation above, we are using the fact that $C^{r+k}_{(b)}(U)$ is a vector space to identify $D\mathcal{T}_f(\delta f) \in T_{\mathcal{T}(f)} C^{r+k}_{(b)}(U)$ with an element of $C^{r+k}_{(b)}(U)$.

$$\begin{aligned} D\mathcal{T}_f(\delta f)(u) &= \tfrac{d}{dt}(c + f + t\delta f)(u, \psi(u, f + t\delta f))\Big|_{t=0} \\ &= \delta f(u, \psi(u, f)) + D_2(c + f)_{(u,\psi(u,f))} \circ D_2\psi_{(u,f)}(\delta f). \end{aligned}$$

Both instances of D_2 in this formula refer to the differential of a function of two arguments with respect to the second argument. But ψ is defined by the equation $D_2(c + f)_{(u,\psi(u,f))} = 0$, and hence the second term vanishes. $\square$

Proposition 22 allows us to make a more specific criterion for abundance. For $f \in \mathcal{F}$, let $\Psi_f : U \to U \oplus V$ denote the map $\Psi_f(u) = (u, \psi(u, f))$. Then Proposition 22 can be restated as $D\mathcal{T}_f(\delta f)(u) = \delta f \circ \Psi_f(u)$.

Corollary 23 *$(\mathcal{T}, \mathcal{F})$ is kth order abundant at (u, f) if and only if the map $\delta f \in T_f\mathcal{F} \mapsto j^k_u(\delta f \circ \Psi_f) \in T_{j^k_u(\mathcal{T}(f))} J^k(U)_u = J^k(U)_u$ is surjective. Note that Proposition 22 implies that we can also write*

$$\mathcal{F}^{(l)}_{(f,u)} = \{\delta f \in T_f\mathcal{F} \mid j^l_u(\delta f \circ \Psi_f) = 0\},$$

and then the abundance condition can be recast as the condition that

$$(d\Psi_f)^* \circ d^l(\cdot)_u : \mathcal{F}^{(l-1)}_{(f,u)} \to S^l(U)$$

is surjective for all $1 \le l \le k$. In particular, when $u = 0$, abundance at $(0,0)$ is equivalent to the surjectivity of the map $i^* \circ d^l(\cdot)_0 : \mathcal{F}^{(l-1)}_{(0,0)} \to S^l(U)$, *where* $i^* : S^l(U \oplus V) \to S^l(U)$ *denotes the pull-back by the inclusion* $i : U \to U \oplus \{0\} \subset U \oplus V$.

We next provide a sufficient condition for $(\mathcal{T}, \mathcal{F})$ to be abundant on a fixed neighborhood $B_U \times B_{\mathcal{F}}$ (assumed to be small enough that ψ is defined on it, of course). In this theorem, let $\Pi_U : U \oplus V \to U$ denote projection onto the first factor. We'll denote the pull-back map on symmetric tensors by $\Pi_U^* : S^k(U) \to S^k(U \oplus V)$.

Theorem 24 *Fix* $(u_0, f_0) \in B_U \times B_{\mathcal{F}}$ *and let* $v_0 = \psi(u_0, f_0)$. *If* $\mathcal{F} \subset C^{r+k+1}(U \oplus V)$ *has the property that the image of the linear map* $d^l(\cdot)_{(u_0,v_0)} : T_f\mathcal{F} \to S^l(U \oplus V)$ *contains* $\pi^* S^l(U)$ *for each* $0 \le l \le k$, *then* $\mathcal{T} : \mathcal{F} \to C^{r+k}(U)$ *defined by equation (4) is* k*th order abundant at* (u_0, f_0).

The proof of Theorem 24 relies, at its heart, on nothing more than some chain rule calculations. Because of compositions and implicitly defined functions, however, the notation is rather daunting. The following lemma helps to distill one of the calculations down a reasonable amount of notation.

Let $h : U \oplus V \to \mathbb{R}$ and $a : U \to V$ be functions (assumed to be sufficiently differentiable that the following derivatives make sense). Define $b : U \to \mathbb{R}$ by $b(u) = h(u, a(u))$.

Lemma 25 *Let* h, a, *and* b *be functions related as above. Suppose that* $d^l h_{(u,a(u))} = 0$ *for* $l < k$ *and that* $d^k h_{(u,a(u))} = \Pi_U^* \tau \in S^k(U \oplus V)$, *for some* $\tau \in S^k(U)$. *Then* $d^l b_u = 0$ *for all* $l < k$ *and* $d^k b_u = \tau$.

PROOF. Successive applications of the chain rule show

$$
\begin{aligned}
db_u(\vec{u}_1) &= D_1 h_{(u,a(u))}(\vec{u}_1) + D_2 h_{(u,a(u))} \circ da_u(\vec{u}_1), \\
d^2 b_u(\vec{u}_1, \vec{u}_2) &= D_1^2 h_{(u,a(u))}(\vec{u}_1, \vec{u}_2) + D_1 D_2 h_{(u,a(u))}(\vec{u}_1, da_u(\vec{u}_2)) \\
&\quad + D_1 D_2 h_{(u,a(u))}(\vec{u}_2, da_u(\vec{u}_1)) + D_2^2 h_{(u,a(u))}(da_u(\vec{u}_1), da_u(\vec{u}_2)) \\
&\quad + D_2 h_{(u,a(u))}(d^2 a_u(\vec{u}_1, \vec{u}_2)),
\end{aligned}
$$

and, in general,

$$
d^l b_u(\vec{u}_1, \ldots, \vec{u}_l) = D_1^l h_{(u,a(u))}(\vec{u}_1, \ldots, \vec{u}_l) + \text{other terms}.
$$

All of the other terms involve $D_1^{l_1} D_2^{l_2} h_{(u,a(u))}$ evaluated on combinations of l_1 of the vectors $\vec{u}_i$ and l_2 tangent vectors to V which arise from evaluating derivatives of a on combinations of the remaining $\vec{u}_i$. Furthermore, in all these other terms, $l_1 + l_2 \le l$ and $l_2 > 0$, so $l_1 < l$.

It follows that if $d^l h_{(u,a(u))} = 0$ then $d^l b_u = 0$. Furthermore, the assumptions of the lemma imply that $d^k b_u = D_1^k h_{(u,a(u))} = \tau$ because all the other terms vanish in the chain rule formula for $d^k b_u$. $\square$

Proof of Theorem 24. We'll rely on the criterion for abundance provided by Proposition 18. Suppose that $\delta f \in T_f \mathcal{F}$ has the properties that δf vanishes to order $l-1$ at $(u, \psi(u,f))$ and that $d^l \delta f_{(u,\psi(u,f))} = \Pi_U^* \tau \in S^l(T_u U)$. Then Proposition 22 and Lemma 25 imply that $D\mathcal{T}_f(\delta f)$ vanishes to order $l-1$ at $(u, \psi(u,f))$ and

$$
\begin{aligned}
d^l \left(D\mathcal{T}_f(\delta f) \right)_u &= d_u^l \left(\delta f(u, \psi(u,f)) \right) \\
&= \Pi_U^* \tau.
\end{aligned}
$$

$\square$

For reasons that will be explained in Section 3, Theorem 24 is of limited use, although a recent development by Mrowka offers some promise that we may get to the point of being able to construct a class of perturbation functions where this hypothesis is verifiable. In the meantime we can use the following theorem, with a weaker hypothesis, which will be shown to hold for the traditional class of "holonomy perturbations" of Taubes and Floer (if the perturbation loops are chosen with a common base point). We'll discuss the relative merits of Theorems 24 and 26 for holonomy perturbations in 3-dimensional gauge theory in greater detail in the next section.

Consider the inclusion $i : U \to U \oplus V$, $i(u) = (u, 0)$ and the pull-back $i^* : S^l(U \oplus V) \to S^l(U)$ on symmetric tensors.

Theorem 26 *Suppose that $\mathcal{F} \subset C^{r+k+1}(U \oplus V)$ has the property that, for each $0 \leq l \leq k$, there exist*

$$
\delta f_1^l, \ldots, \delta f_{m_l}^l \in \mathcal{F}_{(0,0)}^{(l-1)}
$$

with the property that

$$
\left\{ i^* d^l \left(\delta f_1^l \right)_{(0,0)}, \ldots, i^* d^l \left(\delta f_{m_l}^l \right)_{(0,0)} \right\}
$$

span $S^l(U)$. Then $(\mathcal{T}, \mathcal{F})$ is kth order abundant on some neighborhood $B'_U \times B_{\mathcal{F}}$ of $(0,0)$.

PROOF. The graphing map $\Psi : U \to U \oplus V$, $u \mapsto (u, \psi(u,0))$, satisfies $(d\Psi_0)_* = i_*$ and hence $d\Psi_0^* = i^*$ on symmetric tensors.

Choose δf_i^l as in the statement of the theorem, for $0 \le l \le k$. Then

$$d^j \left(D\mathcal{T}_0(\delta f_i^l)\right)_u = d^j \left(\delta f_i^l \circ \Psi\right)_u$$
$$= \begin{cases} 0 & j < l \\ (d\Psi_0)^* d^l(\delta f_i^l)_{(0,0)} & j = l. \end{cases}$$

Abundance at $(0,0)$ follows. Since the definition of abundance concerns surjectivity of a certain differential, it is an open condition in $(f,u) \in \mathcal{F} \times U$. $\square$

In Section 6, we will discuss abundance of the holonomy perturbations in the gauge theory context. The cases where higher order abundance is needed involve structure groups of rank larger than $SU(2)$ or 3-manifolds which are not homology 3-spheres. In either case, the existence of reducible flat connections complicates the analysis. To handle these complications, we must generalize the discussion above to consider a compact Lie group H acting on E and F, such that c and the functions $f \in \mathcal{F}$ are H invariant. Basically, all function spaces such as $C^r(U)$ must then be replaced by the subspace of invariant functions, and corresponding modifications of the jet spaces must be made. Before we do this, we digress to discuss the hypotheses of Theorems 24 and 26 in the gauge theory context.

3 General Discussion of 3-D Gauge Theory Applications of the Jet Transversality Theorem

We pause at this time to explain why this paper contains so much work aimed at proving abundance theorems in gauge theory, and to discuss the relative merits of Theorems 24 and 26 in this setting. In this context, a primary object of study is the moduli space of flat connections (on a fixed principal bundle over a 3-manifold). Flat connections are the critical points of the Chern-Simons functional, and the flat moduli space is modeled on the zero set of a Fredholm exact 1-form. See Section 6 for details.

First of all, consider $G = SU(2)$ connections on integral homology spheres. The trivial connection is the only reducible flat $SU(2)$ connection, and this is isolated, so the reducible connections do not play a big role in the study of the flat moduli spaces (in particular, in Taubes' gauge theoretic description of the Casson invariant in [20]). In other words, although orbits of reducible connections give rise to singularities in the space of connections modulo gauge, these quotient singularities do not present analytical difficulties in studying the topology of the moduli space or how it changes under perturbation. The transversality problems are fairly standard and do not involve equivariant transversality problems. Therefore, considering abundance of order higher than one is unnecessary (and the jet transversality results for $k = 1$ are more easily set up and stated by skipping the whole jet formalism).

There is a standard class of admissible functions by which one perturbs the Chern-Simons functions to render its critical set a nondegenerate moduli space. It is well-known that this is first order abundant at the irreducible connections (see [20] and [7]), and so the existence of generic perturbations producing nondegenerate 0-dimensional moduli spaces follows from standard arguments. Similarly, connecting two such perturbations by a generic path produces a cobordism between the two moduli spaces.

Suppose, however, that one wished to prove that the moduli spaces change only by a standard normal form Morse birth/death, as in Corollary 17, when the perturbation is varied along a generic path. This requires a higher degree of abundance. It is far from clear that the hypothesis of Theorem 24 holds for holonomy perturbations–we can make de Rham cohomological arguments about the derivatives of holonomy in the $U = H^1_A(X; adP)$ directions, but we generally know little about the derivative in the other directions in the slice at A. Specifically, the complementary space V is $(\ker d^*_A \cap \Omega^1(X; adP)) \cap (H^1_A(X; adP)^\perp$. Topological methods do not provide adequate information about derivatives of holonomy in these directions.

In contrast to Theorem 24, the hypothesis of Theorem 26 is a weaker condition and therefore requires less complete knowledge to verify it for a class of admissible functions, and indeed we will verify it for holonomy perturbations defined using loops with a common basepoint. This, combined with some arguments involving openness of the abundance condition and compactness of the (unperturbed) flat moduli space, would allow us to prove that generic paths only have standard Morse births/deaths in the moduli space (with $SU(2)$ connections on integral homology spheres, so there are no reducibles to worry about).

The primary focus of this paper is the gauge theory setting for larger structure groups, where there are multiple strata of reducible flat connections. These connections have nontrivial isotropy, and we are forced to consider instead of the usual jet space a space of jets of invariant functions. Unfortunately, kth order abundance is no longer an open condition. Overcoming this difficulty is the subject of Section 5.

In the literature on this subject, the case of reducible $SU(3)$ connections with $U(1)$ stabilizer was addressed in [3]. Transversality arguments there concerning bifurcations where irreducible perturbed flat connections suddenly pop out of reducible perturbed flat connections required second order abundance (which will be defined in Section 4). An argument given in these papers claims that we can find a disjoint collection of curves such that linear combinations of traces of holonomies around these curves form a second order abundant space of perturbation functions. This argument contains an error, namely a mistaken assumption about representing homology classes with coefficients in a flat twisted bundle by curves with covariantly constant lifts in the flat coefficient bundle. The present paper corrects this error, and also establishes the new equivariant transversality results for $U(3)$ gauge theory on rational homology 3-spheres, and for $SU(4)$ gauge theory on integral homology spheres.

In summary, the rest of this paper will be devoted to the following tasks. We will make the equivariant transversality and abundance issues precise in Section 4. In Section 5, we prove certain "extension theorems" to replace the openness of the abundance condition, which no longer holds in the equivariant case. Finally, we verify that (common-basepoint) holonomy perturbations are abundant in Section 6.

We finish this digression by noting that, if a function on a finite dimensional manifold is not Morse, the smallest space of perturbations that would allow one obtain transversality results can be constructed as follows. Cover the critical set by Kuranishi model neighborhoods, then take a suitable class of functions on the Kuranishi manifolds (parameterized by Ψ), and bump off in both the U and V directions. One can verify that such a class of perturbations satisfies the hypothesis of Theorem 24, and is generally much smaller than a class of functions that is abundant on the entire manifold.

Bump functions, however, are not compatible with the usual analytical framework for gauge theory in dimension three. That framework involves taking Sobolev completions of the space of connections and the gauge group (L^2_p and L^2_{p+1}, for some $p > \frac{1}{2}$, respectively). This is the range for which the standard slice theorem holds, but in this range bump functions destroy the Fredholm properties of the flatness equation. If $f : L^2_p\Omega^1(X; adP) \to \mathbb{R}$ is a bump function applied to the L^2_p distance to A, then df_{A+a} is in L^2_{-p}, which has lower regularity than $d\,\mathrm{cs}(A + a) = -\frac{1}{4\pi}F(A + a)$.

A recent result of Mrowka [18] is a slice theorem in the borderline setting, where the gauge group has $L^2_{\frac{3}{2}}$ Sobolev completion, the space of connections has $L^2_{\frac{1}{2}}$ completion, and hence the curvature (the differential of the Chern-Simons function) is a map from $L^2_{\frac{1}{2}}$ to $L^2_{-\frac{1}{2}}$. This choice of Sobolev norms has the great advantage that the differential of a bump function applied to the $L^2_{\frac{1}{2}}$ distance from a fixed connection A takes values in $L^2_{-\frac{1}{2}}$ just like the curvature map. Using bump functions, one can construct functions where it is possible to control derivatives not only in the $H^1_A(X; adP)$ directions, but also in the complementary directions. There are, however, still other analytical questions remaining to be worked out in the borderline context, for example Lemma 60.

We will discuss holonomy perturbations in detail in Section 6, but first we generalize the present discussion of abundance to the equivariant setting.

4 Symmetry, abundance and the Kuranishi picture

To begin this section, we return to the general setting of a Fredholm map $\zeta : E \to F$ between Banach spaces. Suppose that H is a compact Lie group acting on E and F linearly, preserving the norms, such that ζ is equivariant. Again, we assume that $\zeta(0) = 0$ and study the zero set of ζ near $0 \in E$ (everything we say here applies equally well to any other zero of ζ, with appropriate shifting of the maps). The Kuranishi method works just as before, but now there are additional invariance/equivariance properties of the maps and neighborhoods. Namely, the subspaces $U, V \subset E$ may be taken to be H invariant subspaces, and likewise for $W, V' \subset F$. The map P defined in Section 2.1 is H equivariant, and from this it follows that the maps ψ, Φ and Ψ are equivariant. The neighborhoods B_U and B_V can be taken to be H invariant.

Now assume once again that $\zeta : E \to F \subset E^*$ is the differential of a function $c : E \to \mathbb{R}$, and we now assume that c is H invariant. It follows that $\zeta = dc$ is H equivariant. Of the perturbations $f \in \mathcal{F}$ we will also require H invariance, so we henceforth assume that $\mathcal{F} \subset C_H^{r+k+1}(U \oplus V)$.

We will begin by describing the appropriate modification of the abundance definition for the equivariant setting. Note that if the H action on U is non-trivial, then the set of jets of invariant functions is a proper subset of the space of all jets. H acts on $J^k(U)$, but not in a way that preserves fibers. The role of the jet space $J^k(U)_u$ will be played instead, in the equivariant setting, by $\mathcal{J}_H^k(U)_u = C_H^k(U)/\sim$, where the equivalence $\sim$ reflects the usual notion of equivalence up to order k. (Note that a function that is H invariant could be equivalent to one that is not–hence we use this relation simply to partition $C_H^k(U)$.) We will be able to give a more useful characterization of this space after reviewing some standard results about groups acting on manifolds.

4.1 Local structure of a manifold with a compact group action

We assume for the following three theorems that H is a compact Lie group, acting linearly on a vector space U (though the theorems generalize to actions on manifolds). Note that since H is compact, the action of H is necessarily a proper action. For $u \in U$, let $\mathcal{O}_u$ denote the orbit of u.

Theorem 27 (Slice Theorem (2.3.3) [9]) *Through any point $u_0 \in U$ there exists a slice manifold, by which we mean a submanifold S such that*

(1) $T_{u_0}U = T_{u_0}S \oplus T_{u_0}\mathcal{O}_u$ and for all $u \in S$, $T_uU = T_uS + T_u\mathcal{O}_u$,
(2) S is H_{u_0} invariant, and
(3) if $u \in S$, $h \in H$, and $hu \in S$, then $h \in H_{u_0}$.

In fact, when U is a vector space, the slice manifold can be taken to be an open neighborhood in an affine subspace. (See Subsection 4.2.)

Theorem 28 (Tube Theorem (2.4.1) [9]) *Let u_0 be a point in U. Then there exists an H-invariant neighborhood N of u_0 in U such that the H action on N is equivalent to the action of H on $H \times_{H_{u_0}} B$, where B is an open H_{u_0} invariant neighborhood of 0 in $T_{u_0}U$ modulo the orbit tangent space. Here the stabilizer H_{u_0} acts on the tangent space via the linearization of the action of H_{u_0} on U.*

The next theorem identifies the space of H invariant functions on an invariant neighborhood of u in U with the space of H_u invariant functions on the slice.

Theorem 29 (See Chapter 2.5, [9]) *Let $S \subset U$ be a slice manifold through $u_0 \in U$, i.e. S is an H_{u_0} invariant submanifold, $T_{u_0}U = T_{u_0}S \oplus T_{u_0}\mathcal{O}_{u_0}$, and for any $u \in S$ and $h \in H$ such that $hu \in S$, $h \in H_{u_0}$. Then HS is an H invariant neighborhood of u_0 restriction to S gives an identification $C^k_H(HS) \cong C^k_{H_{u_0}}(S)$.*

The fourth result we need on the local structure of H manifolds concerns a stratification of U into connected components of orbit types. This will be important for us, primarily because $\mathcal{J}^k_H(U)$ fails to be a vector bundle (the fiber dimension varies), but the restriction of $\mathcal{J}^k_H(U)$ to a stratum is a vector bundle. Hence our transversality arguments in the equivariant setting will be worked out stratum by stratum. To state the fourth result, we need the following definition.

Definition 30 *We say two points u and u' in U have the same* orbit type *if there is an H equivariant diffeomorphism from $\mathcal{O}_u$ to $\mathcal{O}_{u'}$.*

Remark 31 *If u and u' have the same orbit type, it follows that H_u and $H_{u'}$ are conjugate subgroups of H. See Lemma 2.6.2 of [9].*

Theorem 32 (Theorem 2.7.4, [9]) *The connected components of orbit types form a Whitney stratification of U. In other words, they partition U into a finite union of submanifolds (which we call strata) $M_i, i \in I$, with the following properties:*

(1) *For each index* $i \in I$*, there is a subset* $I_i \subset I - \{i\}$ *such that the closure of* M_i *in* U *is equal to* $M_i \cup \bigcup_{j \in I_i} M_j$*, and* $\dim M_j < \dim M_i$ *for each* $j \in I_i$*.*
(2) *For each* $i \in I$ *and* $j \in I_i$*, and each sequence* $\{u_n\}$ *in* M_i *such that* $\lim_{n\to\infty} u_n = u \in M_j$ *and* $\lim_{n\to\infty} T_{u_n} M_i = L$ *in the Grassmann bundle of* U*, we have* $T_u M_j \subset L$*.*
(3) *If* u_n *is a sequence as in (2) and* u'_n *is a sequence in the limit stratum* M_j *so that* $\lim_{n\to\infty} u'_n = u$ *then the limit of the 1-dimensional subspaces* $\mathbb{R} \cdot \lambda(u_n, u'_n)$ *of* T_{u_n} *as* $n \to \infty$ *is contained in* L*. Here* λ *is any diffeomorphism of an open neighborhood of the diagonal in* $U \times U$ *with a neighborhood of the zero section of* TU*.*

4.2 Invariant jet spaces

We now discuss other interpretations of the invariant jet space. Let S and S' be two slices through u. By shrinking S and S' if necessary, we can assume there is a diffeomorphism $\phi : S \to S'$ given by $\phi(s) = h(s) \cdot s$ for some function $h : S \to H$. Note that the following diagram commutes, where r_S and $r_{S'}$ denote restrictions to the slices.

$$\begin{array}{ccc} & & C^k_{H_u}(S) \\ & \overset{r_S}{\nearrow} & \\ C^k_H(U) & & \phi^* \uparrow \\ & \overset{r_{S'}}{\searrow} & \\ & & C^k_{H_u}(S') \end{array}$$

Proposition 33 *The (linearized) action of* H_u *on* T_uU *leaves the orbit tangent space* $T_u\mathcal{O}_u$ *invariant, so we obtain an* H_u *equivariant identification* $T_uU \cong T_u\mathcal{O}_u \oplus (T_uU/T_u\mathcal{O}_u)$*. If an* H *invariant inner product is chosen on* U*, then the orthogonal complement* $N_u = (T_u\mathcal{O}_u)^\perp$ *is also* H_u *invariant, and is identified with the quotient space* $T_uU/T_u\mathcal{O}_u$*.*

If $f \in C^k_H(U)$ *vanishes to order* $k-1$ *at* $u \in U$ *(i.e.* $j^{k-1}_u f = [u, 0]$*), then*

$$d^k f_u \in \pi^* \left(S^k_{H_u}(T_uU/T_u\mathcal{O}_u) \right) \subset S^k(T_uU),$$

where $\pi : T_uU \to T_uU/T_u\mathcal{O}_u$ *is projection and* π^* *denotes the pull-back.*

PROOF. Let $\vec{u} \in T_uU$ and $\vec{\gamma} \in T_u\mathcal{O}_u$. Choose a path $u(t)$ with $u(0) = u$ and $u'(0) = \vec{u}$, and choose a path $h(t) \in H$ so that $h(0) = 1$ and $\frac{d}{dt} h(t)u|_{t=0} = \vec{\gamma}$. Then $\frac{d}{dt} h(t)u(t) = \vec{u} + \vec{\gamma}$.

Let $g \in H_u$. Then

$$\begin{aligned}
g_*(\vec{\gamma}) &= \frac{d}{dt} g(h(t)u)\Big|_{t=0} \\
&= \frac{d}{dt}(gh(t)g^{-1})(gu)\Big|_{t=0} \\
&= \frac{d}{dt}(ad_g h(t))u\Big|_{t=0},
\end{aligned}$$

which is also an orbit tangent vector. This proves that $T_u\mathcal{O}_u$ is H_u invariant. The orthogonal complement N_u (with repect to any H_u invariant inner product on T_u) will automatically also be invariant.

If $f \in C^k_H(U)$ and f vanishes to order $k-1$ at u, then most of the terms in the kth order chain rule vanish and we have

$$\begin{aligned}
d^k f_u(\vec{u}+\vec{\gamma}, \ldots, \vec{u}+\vec{\gamma}) &= \frac{d^k}{dt^k} f(h(t)u(t))\Big|_{t=0} \\
&= \frac{d^k}{dt^k} f(u(t))\Big|_{t=0} \\
&= d^k f_u(\vec{u}, \ldots, \vec{u}).
\end{aligned}$$

□

We henceforth use the following method to determine all slice parameterizations at points in the H vector space U with invariant inner product. For each u, we identify $N_u = (T_u\mathcal{O}_u)^\perp$. Then the canonical slice manifold parameterization at u is

$$\phi_u : B \subset N_u \to U, \phi_u(v) = u + v, \tag{5}$$

where B is a neighborhood of $0 \in N_u$.

Remark 34 *There are several useful identifications that can be made concerning equivariant jet spaces. First, if we let W denote the H equivariant neighborhood HS of u, then the restriction map $r_W : C^k_H(U) \to C^k_H(W)$ induces an isomorphism $\mathcal{J}^k_H(U)_u \cong \mathcal{J}^k_H(W)_u$. Also, note that the H_u equivariant diffeomorphism in equation (5) induces an isomorphism $\mathcal{J}^k_{H_u}(N_u)_0 \cong \mathcal{J}^k_{H_u}(S)_u$. Finally, note that the identification between $C^k_H(W)$ and $C^k_{H_u}(S)$ provided by Theorem 29 induces an isomorphism between $\mathcal{J}^k_H(W)_u$ and $\mathcal{J}^k_{H_u}(N_u)_0$. With these identifications, abundance questions at any point u in a H vector space U reduce to questions about abundance at zero in an H_u vector space N_u.*

The main issue complicating jet transversality in the equivariant setting is that $\mathcal{J}^k_H(U)$ does not form a vector bundle over U (the dimension of the fiber depends on the orbit type of u). For a single stratum M_i in the orbit type stratification of Theorem 32, however, $\mathcal{J}^k_H(U)|_{M_i} = \alpha^{-1}(M_i) \subset \mathcal{J}^k_H(U)$ is a vector bundle over M_i, with fiber isomorphic to $S^{0+\cdots+k}_{H_u}(N_u)$, where $u \in M_i$.

Finally, we define abundance in the equivariant setting. Let $C^{r+k}_{(b),H}(U)$ denote the space of H invariant $C^{r+k}_{(b)}$ functions on U.

Definition 35 *Given a compact Lie group H, a finite dimensional H vector space U, a Banach space $\mathcal{F}$, and a C^r map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$, then we will say that $(\mathcal{T}, \mathcal{F})$ is* kth order abundant *at (u, f) if the differential of the map $f \in \mathcal{F} \mapsto j^k_u \circ \mathcal{T}(f) \in \mathcal{J}^k_H(U)_u$, namely $j^k_u \circ D\mathcal{T} : T_f\mathcal{F} \to \mathcal{J}^k_H(H)_u$, is surjective.*

The following proposition is proved by exactly the same argument that was used for Proposition 21. Let U_H denote the set of points fixed under the group action,

$$U_H = \{u \in U \mid hu = u \text{ for all } h \in H\}.$$

Proposition 36 *If $\phi : U \to U$ is an H equivariant $C^{r+k}_{(b)}$ diffeomorphism fixing $u \in U_H$, then any C^r map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ is kth order abundant at (u, f) if and only if $\phi^*\mathcal{T}$ is kth order abundant at (u, f).*

In the next subsection, we discuss equivariant jet transversality along U_H. In Subsection 4.4, we'll extend these results to the other strata, so the following subsection should be viewed as a warm-up for the general case.

4.3 Equivariant transversality on the U_H stratum

In this subsection, we prove equivariant versions of several transversality results in Section 2.4 for the fixed point stratum U_H.

Theorem 37 *Let $A \subset \mathcal{J}^k_H(U)|_{U_H}$ be a submanifold, transverse to the fibers, and fix $r > \max\{0, \dim U_H - \operatorname{codim} A\}$. (Here, $\operatorname{codim} A$ refers to the codimension in $\mathcal{J}^k_H(U)|_{U_H}$.) Suppose $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ is a C^r map. If $(\mathcal{T}, \mathcal{F})$ is kth order abundant at each $(u, f) \in U_H \times \mathcal{F}$, then*

$$\{f \in \mathcal{F} \mid j^k \circ \mathcal{T}(f) : U_H \to \mathcal{J}^k_H(U)|_{U_H} \text{ is transverse to } A\}$$

is residual in $\mathcal{F}$ and hence dense. For f in this residual subset,

$$\{u \in U_H \mid j^k_u(\mathcal{T}(f)) \in A\}$$

is a submanifold of U_H of codimension equal to $\operatorname{codim} A$.

Again, the restriction on r is not a serious one for us, because in applications to the study of critical points in U_H, A will always be a subset of $\{[u, f] \mid u \in U_H, df_u = 0\}$, which has codimension equal to $\dim U_H$ in $\mathcal{J}^k_H(U)|_{U_H}$.

PROOF. The fact that A is transverse to the fibers of $\alpha : \mathcal{J}^k_H(U)|_{U_H} \to U_H$, and the fact that

$$D\left(j^k_u \circ \mathcal{T}\right)_f = j^k_u \circ D\mathcal{T}_f : T_f\mathcal{F} \to T_{j^k(\mathcal{T}(f))_u}\left(\mathcal{J}^k_H(U)|_{U_H}\right) = \mathcal{J}^k_H(U)|_{U_H}$$

is surjective for each $u \in U_H$, together imply that the map $Q : \mathcal{F} \times U_H \to \mathcal{J}^k_H(U)|_{U_H}$, defined by $Q(f, u) = j^k_u(\mathcal{T}(f))$, is transverse to A. Now the proof of Theorem 9 works without modification to complete the argument. $\square$

Corollary 38 *If $(\mathcal{T}, \mathcal{F})$ is kth order abundant for $k \geq 1$, then*

$$\{f \in \mathcal{F} \mid \mathcal{T}(f)|_{U_H} \text{ is Morse}\}$$

is residual in $\mathcal{F}$.

PROOF. We apply Theorem 37 to the set

$$A = \{[u, f] \in \mathcal{J}^k_H(U)|_{U_H} \mid d(\mathcal{T}(f))_u = 0\}.$$

$\square$

We will prove a stronger version of this statement shortly, but the proof requires some knowledge of the invariant 2-jets. For this reason, we recall some elementary representation theory.

Since H is a compact Lie group acting linearly on U, we can decompose U into isotypical subspaces $U = \bigoplus_{i=1}^m \widetilde{U}_i$. Here, *isotypical* means that each irreducible invariant subspace of $\widetilde{U}_i$ is isomorphic to a fixed irreducible representation U_i (and $U_i \not\cong U_j$ for $i \neq j$). In other words, U is isomorphic (as a representation of H) to $\bigoplus_{i=1}^m U_i \otimes_{\mathbb{R}} \underline{\mathbb{R}}^{r_i}$, where H acts irreducibly on U_i and trivially on $\underline{\mathbb{R}}^{r_i}$. (If $U_H \neq 0$, then one of the irreducible representations is the trivial representation.)

Let $F \in S^2_H(U)$. Then F is diagonalizable over $\mathbb{R}$ and each eigenspace is an invariant subspace (this completely characterizes the space of invariant symmetric bilinear forms). In particular, $\ker F = \oplus_{i=1}^m K_i$, where $K_i \subset \widetilde{U}_i$ is an invariant subspace. It follows that $K_i \cong U_i \otimes \underline{\mathbb{R}}^{k_i}$, for some nonnegative integers $k_i \leq r_i$. Define

$$S^2_{H,(k_1,\ldots,k_m)}(U) = \{F \in S^2_H(U) \mid \dim\ker F \cap \widetilde{U}_i = k_i \dim U_i \text{ for } 1 \leq i \leq m\}. \quad (6)$$

Proposition 39 *$S^2_{H,(k_1,\ldots,k_m)}(U)$ has codimension greater than or equal to $\sum_{i=1}^m k_i$ in $S^2_H(U)$. If $\sum_{i=1}^m k_i = 1$, then $S^2_{H,(k_1,\ldots,k_m)}(U)$ has codimension one.*

PROOF. It follows from the characterization of invariant symmetric bilinear forms above in terms of eigenspaces that $S^2_H(U)$ may be identified with $\bigoplus_{i=1}^m S^2_H(\widetilde{U}_i)$ and then $S^2_{H,(k_1,\ldots,k_m)}(U)$ can be identified with

$$\bigoplus_{i=1}^m S^2_{H,k_i}(\widetilde{U}_i) \cong \bigoplus_{i=1}^m S^2_{H,k_i}(U_i \otimes \underline{\mathbb{R}}^{r_i}) \cong \bigoplus_{i=1}^m S^2_{k_i}(\underline{\mathbb{R}}^{r_i}).$$

Here $S^2_k(\underline{\mathbb{R}}^r)$ denotes the set of symmetric bilinear forms on $\mathbb{R}^r$ (with no group invariance condition) with k dimensional kernel.

The characterization of invariant symmetric bilinear forms above implies the first congruence above. On a fixed isotypical subspace $\widetilde{U}_i = U_i \otimes \underline{\mathbb{R}}^{r_i}$, an invariant symmetric bilinear form is determined by its eigenvalues and its eigenspaces, which are all of the form $U_i \otimes V$, where $V \subset \underline{\mathbb{R}}^{r_i}$ is a subspace. This gives the second equivalence.

We now show that $\operatorname{codim} S^2_{k_i}(\underline{\mathbb{R}}^{r_i})$ has codimension $\frac{k_i^2+k_i}{2}$ in $S^2(\underline{\mathbb{R}}^{r_i})$. For this, we can drop the i subscript.

A dense open subset of $S^2_k(\mathbb{R}^r)$ consists of bilinear forms with all nonzero eigenvalue multiplicities equal to one. This is parameterized by

$$\left\{(x_1,\ldots,x_{r-k}) \in \mathbb{R}^{r-k} \mid x_i \neq x_j \text{ for } i \neq j\right\} \times O(r)/O(k),$$

which has dimension

$$r - k + \frac{r^2-r}{2} - \frac{k^2-k}{2} = \frac{r^2+r}{2} - \frac{k^2+k}{2}.$$

This subset therefore has codimension $\frac{k^2+k}{2}$ in $S^2(\mathbb{R}^r)$. The rest of $S^2_k(\mathbb{R}^r)$ is a union of strata of even larger codimension. □

Corollary 38 gives a sufficient condition to prove that generic functions $\mathcal{T}(f)$ are Morse on U_H. We can now prove a stronger statement about the functions $\mathcal{T}(f)$ along U_H.

Corollary 40 *Assume that a C^r map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ is kth order abundant along $U_H \times \mathcal{F}$ for $k \geq 2$. Then for a residual set of $f \in \mathcal{F}$, $\mathcal{T}(f)|_{U_H}$ is Morse and, furthermore, every critical point $u \in (U_H \cap \operatorname{crit}(\mathcal{T}(f))) = \operatorname{crit}(\mathcal{T}(f)|_{U_H})$ has the stronger Morse property that $d^2\mathcal{T}(f)_u$ is a nondegenerate bilinear pairing on $T_uU \otimes T_uU$. (The fact that $\mathcal{T}(f)|_{U_H}$ is Morse implies only that this pairing on $T_uU_H \otimes T_uU_H$ is nondegenerate.)*

PROOF. For any $f \in \mathcal{F}$, $\mathcal{T}(f)$ is an invariant function, and hence for each $u \in U_H$ the differential $d\mathcal{T}(f)_u \in T_u^*(U)$ is an H invariant cotangent vector. But $(T_u^*(U))_H = T_u^* U_H$, so the differential of $\mathcal{T}(f)$ vanishes at u if and only if the differential of $\mathcal{T}(f)|_{U_H}$ vanishes. In other words, $\mathrm{crit}(\mathcal{T}(f)) \cap U_H = \mathrm{crit}(\mathcal{T}(f)|_{U_H})$.

Define

$$\mathcal{S}_1 = \{[u, f] \in \mathcal{J}_H^k(U)|_{U_H} \mid df_u = 0\}.$$

This subspace has codimension equal to $\dim U_H$ in $\mathcal{J}_H^k(U)|_{U_H}$. Also, define for each $s = 1, 2, 3, \ldots$ and for each m-tuple of nonnegative integers $(k_1, \ldots, k_m)$ summing to s,

$$\mathcal{S}_{1,s}^{(k_1,\ldots,k_m)} = \left\{[u, f] \in \mathcal{J}_H^k(U)|_{U_H} \mid df_u = 0 \text{ and } \dim\left(\ker d^2 f_u \cap \widetilde{U}_i\right) = k_i\right\},$$

and set

$$\mathcal{S}_{1,s} = \bigcup_{(k_1,\ldots,k_m)} \mathcal{S}_{1,s}^{(k_1,\ldots,k_m)}.$$

Each subset $\mathcal{S}_{1,s}^{(k_1,\ldots,k_m)}$ is a submanifold of codimension greater than one in $\mathcal{S}_1$, if $s > 1$, and of codimension one in $\mathcal{S}_1$ if $s = 1$. Furthermore, these submanifolds are transverse to the fibers of $\alpha : \mathcal{J}_H^k(U)|_{U_H} \to U_H$. Thus Theorem 37 provides us with a residual subset of $\mathcal{F}$ for which $j^2 \circ \mathcal{T}(f)$ is transverse to all the submanifolds $\mathcal{S}_{1,s}^{(k_1,\ldots,k_m)}$ and to $\mathcal{S}_1$.

For such an f, the critical set of $\mathcal{T}(f)|_{U_H}$ is the preimage $(j^2(\mathcal{T}(f)))^{-1}(\mathcal{S}_1)$. By the transversality assumption, this is a 0-dimensional submanifold of U_H. Furthermore, the set of critical points $u \in U_H$ for which $d^2(\mathcal{T}(f))_u$ is a degenerate symmetric bilinear pairing is exactly

$$\bigcup_{s \geq 1} (j^2(\mathcal{T}(f)))^{-1}(\mathcal{S}_{1,s}),$$

which has codimension greater than $\dim U_H$ and hence is empty. $\square$

4.4 Equivariant transversality on arbitrary strata

In Subsection 4.3, we considered transversality issues along the bottom stratum U_H of the orbit type stratification. This stratum is simpler than the other strata for the following two reasons:

- U_H is actually a linear subspace of U, not simply a submanifold.
- For each $u \in U_H$, the orbit tangent space is trivial and the slice is U.

In this subsection, we revise the argument so that it will work for any stratum. To do this, we will use the slice parameterization $\phi_u : B \subset N_u \to S \subset U$, where B is an H_u invariant neighborhood of $0 \in N_u$, to reduce the transversality questions on the stratum containing u to transversality questions on the bottom stratum $(N_u)_{H_u} \subset N_u$, to which we can apply Theorem 37 and Corollary 40.

Fix an element $u \in U$ and let M denote the stratum of U containing u. Note that M is an H invariant subset of U. If we fix a parameterization $\phi : B \subset N_u \to S \subset U$ for a slice through u, then ϕ identifies $B \cap (N_u)_{H_u}$ (more precisely, the component of this intersection containing $0 \in N_u$, but we may as well assume that $B \cap (N_u)_{H_u}$ is connected) with $S_{H_u} = M \cap S$, which is a slice for the H action on M.

In particular, setting $U' = N_u$ and $H' = H_u$, we can apply Theorem 37 to this vector space (or to an open invariant neighborhood of the origin in it), obtaining a local result about transversality in $S \cap M$ (in fact, on the the equivariant neighborhood $H(S \cap M)$). Covering M with countably many neighborhoods of this form, we obtain the following global result on the stratum M.

Theorem 41 *Let $u_0 \in U$ and let M denote the connected component of orbit type containing u_0. Let $A \subset \mathcal{J}^k_H(U)|_M$ be a submanifold, transverse to the fibers. Fix $r > \{0, \dim(M/H) - \operatorname{codim} A\}$. Suppose $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ is a C^r map. If $(\mathcal{T}, \mathcal{F})$ is kth order abundant at each $(u, f) \in M \times \mathcal{F}$, then*

$$\{f \in \mathcal{F} \mid j^k \circ \mathcal{T}(f) : M \to \mathcal{J}^k_H(U)|_M \text{ is transverse to } A\}$$

is residual in $\mathcal{F}$ and hence dense. For f in this residual subset,

$$\{u \in M \mid j^k_u(\mathcal{T}(f)) \in A\}$$

is a submanifold of M of codimension equal to $\operatorname{codim} A$.

Corollary 42 *Assume that $\mathcal{T} : \mathcal{F} \to C^{r+k}_H(U)$ is a C^r map which is kth order abundant along $M \times \mathcal{F}$, for $k \geq 2$. Then for a residual set of $f \in \mathcal{F}$, $\mathcal{T}(f)|_M$ is Bott-Morse with isolated critical orbits, and furthermore every critical point has the stronger Morse property that $d^2\mathcal{T}(f)_u$ is a nondegenerate bilinear pairing on $N_u \otimes N_u$.*

From this corollary, we obtain the following theorem. A version of this for an H manifold M and for the space of smooth functions $\mathcal{F} = C^\infty_H(M)$ was first proved by Wasserman in [22].

Theorem 43 *Suppose U is a finite dimensional H vector space, for a compact Lie group H. Furthermore, suppose that $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ is a C^r map which is kth order abundant at each (u, f), for some $k \geq 2$. Then for a residual set of $f \in \mathcal{F}$, $\mathcal{T}(f)$ is Bott-Morse with isolated orbits of critical points.*

PROOF. Corollary 42 shows that, for a fixed stratum, a residual set of $f \in \mathcal{F}$ is Morse restricted to the stratum and furthermore satisfies the stronger Morse property of Corollary 40. The intersection of all these residual subsets of $\mathcal{F}$, indexed by the strata, is a residual subset where the strong Morse property holds on each stratum. The Morse property for restrictions of $\mathcal{T}(f)$ to any slice in the stratum guarantees that the orbits of critical points in the stratum are isolated. The stronger Morse property prevents orbits of critical points in different strata from limiting to this stratum. (Critical points must, of course, occur in whole orbits, since the functions are invariant). $\square$

Remark 44 *On the principal orbit strata, only first order abundance is required. Here H_u acts trivially on N_u, so the nonequivariant argument in Corollary 11 works.*

Remark 45 *We pause to compare Theorem 43 and Wasserman's Theorem. The difference is that Theorem 43 gives a sufficient condition on a nonlinear map $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ to obtain Morse functions $\mathcal{T}(f)$, instead of simply proving that Morse functions form a residual subset of $C^k_{(b),H}(U)$. Thus it can be applied to Kuranishi models arising from Fredholm exact 1-forms. In addition, the present formulation of transversality gives additional information about the parameterized critical set for paths of invariant functions, a subject that was not part of Wasserman's discussion. This parameterized critical set is explored in the next subsection.*

4.5 Bifurcations in the critical set for generic 1-parameter families

In this subsection, we discuss the parameterized critical set for generic 1-parameter families of functions in $\mathcal{F}$. We begin with some terminology.

Suppose that V is an H vector space. As a representation of H, V can be decomposed as

$$V \cong \oplus_{i=1}^m (V_i \otimes \underline{\mathbb{R}}^{r_i}),$$

where the H action on V_i is irreducible. If H acts transitively on the unit sphere in each irreducible representation V_i (with respect to an invariant metric), we say the H action on V has the *sphere transitivity property*. For $u \in U$, let $[u] \in U/H$ denote the orbit of u.

Theorem 46 *Let H be a compact Lie group and U be a finite dimensional H vector space. Suppose that, for each $u \in U$, the H_u action on N_u has the sphere transitivity property.*

Assume that $\mathcal{T} : \mathcal{F} \to C^{r+k}_{(b),H}(U)$ satisfies the hypothesis of Theorem 43. For two functions f_0 and f_1 in $\mathcal{F}$ such that $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$ are Bott-Morse *with isolated critical orbits, let $\widetilde{C}^{r+k}([0,1],\mathcal{F})$ be the space of paths with these endpoints as in Theorem 12. Then for paths $\{f_t\}$ in a residual subset of $\widetilde{C}^{r+k}([0,1],\mathcal{F})$, the following properties hold:*

(1) For any stratum M, the parameterized critical set in M, reduced modulo H, is a C^r 1-manifold of $M/H \times [0,1]$.
(2) For all $t \in (0,1)$ except for a discrete set $\{t_1,\dots,t_m\}$, the function $\mathcal{T}(f_t)$ is Bott-Morse with isolated critical orbits. For each t_i, there are finitely many degenerate critical orbits $[p_1^i],\dots,[p_{m_i}^i],[q_1^i],\dots,[q_{n_i}^i]$ of $\mathcal{T}(f_{t_i})$, and these have the following properties:
   (a) $\ker d^2(f_{t_i})_{p_j^i} \subset N_{p_j^i}$ is an irreducible nontrivial representation of $H_{p_j^i}$, and
   (b) $\ker d^2(f_{t_i})_{q_j^i}$ is a 1-dimensional trivial representation of $H_{q_j^i}$.
(3) Near a point $([q_j^i],t_i)$, there are no other critical points of different orbit types.
(4) Near a point $([p_j^i],t_i)$, the parameterized critical set in the p_j^i stratum (modulo H) is parameterized by $([p_j^i(t)],t)$, $t \in (t_i-\epsilon,t_i+\epsilon)$. Furthermore the eigenvalue of $d^2(\mathcal{T}(f_t))_{p_j^i(t)}$ which crosses zero at t_i does so transversely.
(5) If we stratify the parameterized critical set modulo H by orbit type, the only points in one stratum which are in the closure of another stratum are the points $([p_j^i],t_i)$. Each of these is the limit of one noncompact end of another stratum, which has smaller stabilizer, and the stabilizer of the second stratum is that of the nonzero points in $\ker d^2(\mathcal{T}(f_{t_i}))_{p_j^i}$.

PROOF. Since the space of bilinear forms on N_u with kernel equal to one irreducible subspace (under the H_u actions) has codimension 1, these do occur, for isolated t's, for generic 1-parameter families. This is analogous to Corollary 15, but uses Proposition 39. This proves the first two claims. The third follows from a standard Kuranishi argument, since the Hessian $d^2(f_{t_i})_{q_j^i}$ is nondegenerate on the normal bundle to the stratum.

For the fourth claim, first note that $d^2(f_{t_i})_{p_i^j}$ restricted to (a slice manifold for) the stratum of p_i^j is nondegenerate, from which it follows that there is a differentiable family $p_j^i(t)$ of the same orbit type whose orbits parameterize all of the parameterized critical set near $([p_j^i],t_i)$. The second part of Condition (4) now follows from an argument analogous to the proof of Corollary 15.

The fifth claim requires a Kuranishi argument to reduce the analysis of the parameterized critical set near $([p_j^i],t_i)$ to the study of the critical set of a path of $H_{p_j^i}$ invariant functions on $V=\ker d^2(f_{t_i})_{p_j^i}$. By the sphere transitivity, this reduces to a path of $\mathbb{Z}_2$ invariant functions on a line (restrict the functions on V to any line through the origin). There the change in the critical set is a standard *pitchfork bifurcation.* $\square$

Remark 47 *Again, for the strata of principal orbits, only first order abundance is required, as in Corollary 13.*

5 Abundance extension properties

We noted in Proposition 26 that kth order abundance at (u, f) is an open condition in the nonequivariant case. Such is not the case in the equivariant setting, as we show with this simple example.

Example. Let $H = U(1)$ and let $U = U_H \oplus U_H^{\perp}$ where $U_H \cong \underline{\mathbb{R}}$ and $U_H^{\perp} \cong \mathbb{C}$ with the standard $U(1)$ action. Let $\pi : U \to U_H$ be projection and set $\mathcal{F} = \{\text{constants}\} \oplus \pi^* Hom(U_H, \mathbb{R})$ (so $\mathcal{T} : \mathcal{F} \to C^3_{(b),H}(U)$ is simply the inclusion map). Then $(\mathcal{T}, \mathcal{F})$ is first order abundant at $(0, 0) \in U \times \mathcal{F}$ but not first order abundant at any $u \notin U_H$. Indeed, if $u = (x, z)$ with $z \neq 0$, then no function $f \in \mathcal{F}$ has nonzero derivative in any direction in $\{0\} \oplus U_H^{\perp}$.

This example illustrates one of the challenges of using holonomy perturbations in the gauge theory setting. We will show abundance (to arbitrarily high order) at a flat connection which is the center of a Kuranishi neighborhood. But to develop useful transversality results, we need abundance in an open neighborhood (ultimately, a neighborhood of the flat moduli space, but we get this by covering the compact flat moduli space with a finite number of neighborhoods of flat connections).

Since it is not true that kth order abundance is an open condition, we will prove certain results of the form "kth order abundance at $0 \in U$ implies second order abundance on a neighborhood of $0 \in U$." Second order abundance is all that is necessary for Theorem 43, which provides the existence of Morse functions in $\mathcal{F}$. It is also sufficient for the transversality results concerning parameterized critical sets for paths of functions in Theorem 46. For the $SU(4)$ gauge theory applications in Section 6, all Kuranishi models have the sphere transitivity property, so ultimately second order abundance is sufficient. But to obtain second order abundance on open sets in the Kuranishi models, we need higher order abundance at the flat connections.

How high k must be for kth order abundance at the origin to imply second order abundance near the origin depends on the H representation U, but is not a terribly important issue, since Proposition 62 guarantees that the traditional holonomy perturbations discussed later in this paper can be made abundant to arbitrarily high order at each flat connection. The tricky, but important, thing is to show that for some k this implication holds.

We now offer a roadmap of the representation-specific calculations in this section. Subsection 5.1 discusses the case $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^l$ and $H = U(1)$. This is the simplest case (and the only one required for $SU(3)$ gauge theory on integral homology spheres, as in [3]) and we present a detailed argument in this case.

The remaining subsections cover other representations. Their outlines parallel the argument in Subsection 5.1, although calculations and the statements of the lemmas vary somewhat. In Subsection 5.2, we explain all the modifications necessary to make the arguments in Subsection 5.1 work for the representation $U = \underline{\mathbb{R}}^k \oplus (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l)$ of $H = SO(3)$, appealing to the earlier argument wherever possible. Subsection 5.3 discusses the representation $U = \underline{\mathbb{R}}^k \oplus (\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l)$ of $U(2)$, and the arguments parallel Subsection 5.2 very closely, so this subsection focuses only on the parts that differ significantly. Finally, Subsection 5.4 discusses a representation $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2} \oplus \mathbb{C}^{l_3}$ of $H = U(1) \times U(1) \times U(1)$.

5.1 $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^l$ *and* $H = U(1)$

In the relatively simple case when $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^l$, there are only two orbit type strata. The bottom stratum consists of the fixed points $U_H = \underline{\mathbb{R}}^k \oplus \{0\}$ and the top stratum consists of points with trivial isotropy $H_u = \{1\}$. The purpose of this subsection is to prove the following extension theorem.

Theorem 48 *Let $H = U(1)$ and $U \cong \underline{\mathbb{R}}^k \oplus \mathbb{C}^l$, with H acting by scalar multiplication on $\mathbb{C}^l$. If a C^2 map $\mathcal{T} : \mathcal{F} \to C^4_{(b),H}(U)$ is second order abundant at $(0,0) \in U \times \mathcal{F}$, then there is a neighborhood $B_U \times B_{\mathcal{F}}$ of $(0,0)$ such that for every $(u,f) \in B_U \times B_{\mathcal{F}}$, $(\mathcal{T}, \mathcal{F})$ is*

(i) second order abundant at (u,f) if $H_u = H$, and
(ii) first order abundant at (u,f) if $H_u = \{1\}$.

While Claim (i) follows immediately from the fact that second order abundance is an open condition in $(u,f) \in U_H \times \mathcal{F}$, the proof of Claim (ii) will require a series of lemmas. Let $P^k(U)$ denote the set of polynomials of degree $\leq k$ on U and $P^k_H(U)$ denote the subset of H invariant polynomials.

Lemma 49 *Let $H = U(1)$ and $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^l$. Then $P^2_H(U)$ is first order abundant at each $u = (0,z)$, where $z = (\lambda, 0, \ldots, 0)$ and $\lambda > 0$.*

PROOF. With $u = (0, z)$ as above, $H_u = \{1\}$ and the orbit tangent space is spanned by the single vector $(0, (i, 0, \ldots, 0))$. The normal space N_u to the orbit tangent space is identified with $\underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}} \oplus \mathbb{C}^{l-1}$, where the single $\mathbb{R}$ summand corresponds to the direction spanned by u.

For the following proof, we assume $k = 1$ and $l = 2$. Generalizing to higher dimensions is completely straightforward. We put coordinates (x_1, z_1, z_2) on $U = \mathbb{R} \oplus \mathbb{C}^2$. Denote the real and imaginary components of z_j by a_j and b_j. For two complex numbers z_1 and z_2, let $\langle z_1, z_2 \rangle = \Re(z_1 \overline{z}_2)$.

The H invariant polynomials of degree ≤ 2 on U are

$$P_H^2(U) = span\{1, x_1, x_1^2, |z_1|^2, |z_2|^2, \langle z_1, z_2 \rangle, \langle z_1, iz_2 \rangle\}. \tag{7}$$

We use the slice parameterization $\phi_u : (x_1, a_1, a_2, b_2) \mapsto (x_1, \lambda + a_1, a_2, b_2)$ to obtain coordinates on the slice through u, thereby identifying it with N_u. The $H_u = \{1\}$ invariant polynomials of degree ≤ 1 on N_u are simply

$$P_{H_u}^1(N_u) = P^1(N_u) = span\{1, x_1, a_1, a_2, b_2\}. \tag{8}$$

The map sending each $p \in P_H^2(U)$ to the pull-back $\phi_u^*(p) = p \circ \phi_u$, truncated to a degree 1 polynomial, is given in terms of the bases above by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}. \tag{9}$$

The fact that the third and fifth columns are zero means we do not need the full basis of degree ≤ 2 polynomials to obtain first order abundance at u. In particular, we may as well delete x_1^2 and $|z_2|^2$. Clearly, removing the third and fifth columns results in an invertible matrix when $\lambda \neq 0$. $\square$

The next three lemmas concern subspaces $V \subset C_H^4(U)$ which are second order abundant at $0 \in U$. To relate this setting to the general considerations of $\mathcal{T} : \mathcal{F} \to C_H^{r+k}(U)$ considered in this paper, this is the case when $\mathcal{T}$ is linear and injective. Note that when $\mathcal{T}$ is linear, abundance at (u, f) is independent of f.

We will generalize Lemma 49 in the following ways. First, we show that V is first order abundant for sufficiently small u of the form $(0, (\lambda, 0, \ldots, 0))$ of the form discussed before the lemma. Then we show that the same result holds for all sufficiently small u of the form $u = (0, z) \in \{0\} \oplus \mathbb{C}^l$. Finally, we explain how the entire argument can be improved to work for all nonzero $u \in U$ sufficiently small.

Lemma 50 *Let U and H be as in Lemma 49, and let $V \subset C^4_H(U)$ be a subspace such that V is second order abundant at $0 \in U$. Then V is first order abundant at all points $u = (0, (\lambda, 0, \ldots))$ for $\lambda > 0$ sufficiently small.*

PROOF. Suppose $V \subset C^4_H(U)$ is second order abundant at $0 \in U$. Then we can choose functions $f_1, \ldots, f_5 \in V$ such that (as ordered lists)

$$\{j^2_0 f_1, \ldots, j^2_0 f_5\} = \left\{ j^2_0(1), j^2_0(x_1), j^2_0|z_1|^2, j^2_0\langle z_1, z_2\rangle, j^2_0\langle z_1, iz_2\rangle \right\}.$$

(Again, we are giving the argument for the case $U = \underline{\mathbb{R}} \oplus \mathbb{C}^2$, but generalizing to higher dimensions is trivial.) Let $B \subset U_H^\perp$ be a bounded neighborhood of 0. Then we can find a constant M such that $|d^3 f_i| < M$ on $\{0\} \times B \subset U_H \oplus U_H^\perp$.

Using Taylor's Theorem, one can show that if $f \in C^4_H(U)$ vanishes to order 2 at the origin and $|d^3 f| < M$ at each point in B, then $|f(u)| < M\|u\|^3$ and $|df_u| < M\|u\|^2$. Applying this to the Taylor remainders $\{f_1 - 1, f_2 - x_1, f_3 - |z_1|^2, \ldots\}$, we find that the first order Taylor expansions of the f_i about $u = (0, (\lambda, 0, \ldots, 0))$, are given, in terms of the basis displayed in (8), by the matrix (9) plus another matrix whose entries are bounded by $M\lambda^2$. It follows that, for $\lambda > 0$ sufficiently small (depending on M) that the latter matrix is invertible. In particular, the map sending functions in $span\{f_i\}_{i=1,\ldots,m}$ to their 1-jets at $u = (0, (\lambda, 0, \ldots, 0))$ is surjective. $\square$

Next we address the issue of abundance at points $u = (0, z) \in U_H \oplus U_H^\perp$ where $z \neq 0$ is small but not of the form $z = (\lambda, 0, \ldots, 0)$.

Lemma 51 *Let U and H be as in Lemma 49, and let $V \subset C^4_H(U)$ be second order abundant at $0 \in U$. Then V is first order abundant at all points u of the form $u = (0, z)$ where $z \in \mathbb{C}^l$ is sufficiently small.*

PROOF. Put an $H = U(1)$ equivariant inner product on U, which we take to be the standard one on $\underline{\mathbb{R}}^k \oplus \mathbb{C}^l$. Set $K = U(l)$, and let K act on U by a trivial action on the $\underline{\mathbb{R}}^k$ factor and by the usual action on $\mathbb{C}^l$, so that the action of K commutes with the action of H, and K also preserves the inner product.

Under the identification

$$d^0(\cdot)_0 \oplus \cdots \oplus d^k(\cdot)_0 : P^k_H(U) \xrightarrow{\cong} S^{0+\cdots+k}_H(U) \subset \bigoplus_{j=0}^{k} (U^*)^{\otimes j},$$

the space of polynomials inherits an inner product. (The basis we wrote down for $P^2_H(U)$ was orthogonal, but not not normal.) With respect to this inner product, K acts orthogonally on the space of invariant polynomials (via pull-back $h^*p = p \circ h$). Note also that, for each $g \in K$, the linearization of the diffeomorphism $g : U \to U$ at u is an isometry $dg_u : N_u \to N_{gu}$ with respect to the inner products inherited by N_u and N_{gu} from that on U.

Suppose that $u = (0, (\lambda, 0, \ldots, 0))$, as above, and let $u' = gu$ for some $g \in K$. If $p_1, \ldots, p_m \in P^2_H(U)$ are the polynomials utilized in the proof given for the u special case, then $(g^{-1})^* p_j = p_j \circ g^{-1}$ are a collection of invariant polynomials which play the same roles for u', namely to prove that $P^2_H(U)$ is first order abundant at u'.

If $f_1, \ldots, f_m$ were the functions in V used in the special case, then the functions $g^* f_j = f_j \circ g$ will have 1-jets spanning $\mathcal{J}^1_H(U)_{u'}$, but the problem is that these functions $g^* f_j$ may not be in V. Therefore, we need a different argument to go from the polynomials to functions in V.

Let $\{p_1, \ldots, p_m\}$ be the basis for $P^2_H(U)$ specified in equation (7). Then there are coefficients $c_{i,j}(g), 1 \le i, j \le m$ depending on $g \in K$ such that $g^* p_i = \sum_{j=1}^m c_{i,j}(g) p_j$. If we choose functions $f_1, \ldots, f_m \in V$ representing the jet classes of $p_1, \ldots, p_m$ at 0, then $\sum_{j=1}^m c_{i,j}(g) f_j$ are functions in V with the property that

$$j^2_0 \left(\sum_j c_{i,j}(g) f_j \right) = j^2_0 \left(g^* p_i \right).$$

Since K is a compact group, the coefficients $c_{i,j}$ are uniformly bounded, and we can redo the argument allowing for the bounds on these coefficients. Thus, for a possibly smaller range of positive λ values, V must be first order abundant at all gu, $g \in K$ and $u = (0, (\lambda, 0, \ldots, 0))$. Notice, however, that K acts transitively on the unit sphere in $\mathbb{C}^l$, so this proves the lemma. $\square$

Finally we extend these results concerning $V \subset C^4_H(U)$ to obtain first order abundance on a neighborhood of zero in U.

Lemma 52 *Let U and H be as in Lemma 49, and let $V \subset C^4_H(U)$ be second order abundant at $0 \in U$. Then there exists a neighborhood $B_1 \times B_2 \subset U_H \times U_H^\perp$ of $(0,0)$ such that V is*

(i) second order abundant on $B_1 \times \{0\}$, and

(ii) first order abundant on $B_1 \times (B_2 - \{0\})$.

PROOF. Assume V is second order abundant at $0 \in U$. This is an open condition in $u \in U_H$, so we can find an open neighborhood $B_1 \subset U_H$ such that V is second order abundant at all $u \in B_1 \times \{0\}$.

Then we can choose families of functions $f_1^{x_0}, \ldots, f_m^{x_0}$ parameterized $x_0 \in B_1$ with the property that

$$j^2_{(x_0,0)}\left(f_j^{x_0}(x,z)\right) = j^2_0\left(p_j(x - x_0, z)\right).$$

After possibly shrinking $B_1 \subset U_H$ to a smaller neighborhood of 0, we can choose a neighborhood $B_2 \subset U_H^\perp$ of 0 and a uniform bound M for $d^3_{(x_0,z)} f^{x_0}$ independent of $x_0 \in B_1$ and $z \in B_2$. Then the proof of Lemma 51 works for each fiber of the projection $B_1 \times B_2 \to B_1$, producing a smaller neighborhood $x_0 \times B_2^{x_0}$ for each $x_0 \in B_1$. But since the derivative bounds are independent of x_0, there is a lower bound on the size of $B_2^{x_0}$, independent of x_0. In other words, there is a smaller $B_2' \subset B_2 \subset U_H^\perp$ such that V is first order abundant at all $u = (x,z) \in B_1 \times (B_2' - \{0\})$, which proves the lemma. $\Box$

Proof of Theorem 48. It remains to prove extend Lemma 52 to nonlinear maps. Specifically, if $\mathcal{T} : \mathcal{F} \to C^4_H(U)$ is a nonlinear map which is abundant at $(0,0) \in U \times \mathcal{F}$, then we must establish the existence of a neighborhood $B_{\mathcal{F}} \subset \mathcal{F}$ of 0 such that the $(\mathcal{T}, \mathcal{F})$ is abundant at (u, f) (to the appropriate order, depending on the orbit type of u) for all $f \in B_{\mathcal{F}}$ and for all u in some neighborhood $B_1 \times B_2 \subset U_H \times U_H^\perp$ of $(0,0)$.

Notice that Lemma 52 allows us to find a neighborhood $B_1 \times B_2 \subset U_H \times U_H^\perp$ such that the desired abundance statements in the theorem hold at all $(u, 0) \in (B_1 \times B_2) \times \{0\} \subset U_H \times U_H^\perp \times \mathcal{F}$. (In place of the functions $f_j^{x_0} \in V$ chosen in Lemma 52, we choose tangent vectors $(\delta f)_j^{x_0} \in T_0\mathcal{F}$.)

To prove the theorem, we choose parameterized families $(\delta f)_j^{(x_0,f)}$ parameterized by $(x_0, f) \in U_H \times \mathcal{F}$ near $(0,0)$. This is possible because second order abundance at $((x_0, 0), f)$ is open in $x_0 \in U_H$ and $f \in \mathcal{F}$. After possibly shrinking neighborhoods, we can find a uniform bound on all the third derivatives

$$d^3_{(x_0,z)}(\delta f)_j^{(x_0,f)}$$

for $(x_0, z, f) \in B_1 \times B_2 \times B_{\mathcal{F}}$. Then the argument works for the nonlinear case, completing the proof of the theorem. $\Box$

5.2 $U = \underline{\mathbb{R}}^k \oplus (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l)$ *and* $H = SO(3)$

The purpose of this subsection is to prove an analog of Theorem 48 for the representation $U = \underline{\mathbb{R}}^k \oplus (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l)$ of $H = SO(3)$. This time, however, fourth order abundance at the origin is necessary to make the argument work.

Theorem 53 *Suppose that a* C^2 *map* $\mathcal{T} : \mathcal{F} \to C^6_{(b),H}(U)$ *is fourth order abundant at* $(0,0) \in U \times \mathcal{F}$. *Then there exists a neighborhood* $B_U \times B_{\mathcal{F}} \subset U \times \mathcal{F}$ *of* $(0,0)$ *such that for all* $(u,f) \in B_U \times B_{\mathcal{F}}$, $(\mathcal{T}, \mathcal{F})$ *is*

(i) fourth order abundant if $H_u = H$,
(ii) second order abundant if $H_u \cong U(1)$, *and*
(iii) first order abundant if $H_u = \{1\}$.

As in Subsection 5.1, we begin by considering $u \in U_H^\perp$, and as in Lemma 51 we will use a larger symmetry group K acting on U to reduce each nonzero $u \in U_H^\perp$ to a canonical form (actually, here there are two canonical forms, depending on the orbit type of u). Fixing an H invariant inner product on U, we can let $K = SO(l)$ act on $\mathbb{R}^3 \otimes \mathbb{R}^l$, commuting with the action of H and preserving the inner product. In fact, setting $K' = SO(l) \times SO(3)$, we can use K' in the same manner as we used $K = U(l)$ in Subsection 5.1.

Lemma 54 *The space of invariant polynomials* $P^4_H(U)$

(i) is second order abundant at each $u \in U$ *with* $H_u \cong U(1)$, *and*
(ii) is first order abundant at each $u \in U$ *with* $H_u = \{1\}$.

Remark 55 *The proof of Lemma 54 is a direct calculation, given below. As in Section 5.1, to go from abundance for the space of polynomials to the general case, we need not only the lemma as stated but certain bounds on the analog of the matrix (9), but this is awkward to state in a lemma, so we simply establish it in the course of the proof.*

PROOF. Identifying $\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l$ with $(\mathbb{R}^3)^l$, we will call a nonzero $v = (v_1, \ldots, v_l)$ *decomposable* if the vectors $\{v_1, \ldots, v_l\}$ form a linearly dependent set (otherwise we call v *indecomposable*). This terminology comes from the fact that the corresponding element of $\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l$ can be written as a single tensor $e \otimes f$. Under the action of K', any decomposable tensor $v \in \mathbb{R}^3 \otimes \underline{\mathbb{R}}^l$ can be put in the form $v = (\lambda e_1, 0, \ldots, 0)$, and any indecomposable tensor can be put in the form $v = (\lambda e_1, \mu e_2)$ if $l = 2$ or $v = (\lambda e_1, \mu e_2, \nu e_3, 0, \ldots, 0)$ if $l \geq 3$, with $\lambda \geq \mu \geq \nu \geq 0$. To see this, choose unit vectors $e \in \mathbb{R}^3$ and $\bar{e} \in \underline{\mathbb{R}}^l$ so that $\langle v, e \otimes \bar{e} \rangle$ is as large as possible. Choosing these as the first elements in orthonormal bases for $\mathbb{R}^3$ and $\underline{\mathbb{R}}^l$ insures that $v = (\lambda e, v_2, \ldots, v_l)$ and $v_j \perp e$ for $j = 2, \ldots, l$. Then choose the second elements for bases for $\mathbb{R}^3$ and $\underline{\mathbb{R}}^l$

orthogonal to the first, maximizing the next projection. Continuing in this fashion we obtain an orthogonal change of basis (differing by an element of K') after which v has the desired form.

Part (i). The Indecomposable Case ($H_u = \{1\}$).
We will prove the indecomposable case when $k = 0$ and $l = 3$. Generalizing to the other cases is trivial. Let $\lambda \geq \mu > 0$ and $\mu \geq \nu \geq 0$ and set

$$u = (\lambda e_1, \mu e_2, \nu e_3) \in \mathbb{R}^3 \otimes \underline{\mathbb{R}}^3 \cong \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3.$$

The orbit of u is 3-dimensional, and $T_u\mathcal{O}_u$ is spanned by

$$\{(\lambda e_2, -\mu e_1, 0), (\lambda e_3, 0, -\nu e_1), (0, \mu e_3, -\nu e_2)\}.$$

The following is an orthonormal basis for N_u:

$$\{(e_1, 0, 0), (0, e_2, 0), (0, 0, e_3), \tfrac{1}{\sqrt{\lambda^2+\mu^2}}(\mu e_2, \lambda e_1, 0),$$
$$\tfrac{1}{\sqrt{\lambda^2+\nu^2}}(\nu e_3, 0, \lambda e_1), \tfrac{1}{\sqrt{\mu^2+\nu^2}}(0, \nu e_3, \mu e_2)\}.$$

We use this basis to coordinatize the slice $S = u + N_u$ with the map

$$\phi_u(a,b,c,d,e,f) = \Big((\lambda + a)e_1 + \tfrac{d\mu}{\sqrt{\lambda^2+\mu^2}}e_2 + \tfrac{e\nu}{\sqrt{\lambda^2+\mu^2}}e_3,$$
$$\tfrac{d\lambda}{\sqrt{\lambda^2+\mu^2}}e_1 + (\mu + b)e_2 + \tfrac{f\nu}{\sqrt{\mu^2+\nu^2}}e_3, \tfrac{e\lambda}{\sqrt{\lambda^2+\nu^2}}e_1 + \tfrac{f\mu}{\sqrt{\mu^2+\nu^2}}e_2 + (\nu + c)e_3\Big).$$

For $v \in \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$, write $v = (v_1, v_2, v_3)$. Consider the collection of invariant polynomials

$$\Big\{1, \|v_1\|^2, \|v_2\|^2, \langle v_1, v_2\rangle, \langle v_1, v_3\rangle, \langle v_2, v_3\rangle, \det[v_1 v_2 v_3]\Big\} \tag{10}$$

and the basis $\{1, a, b, c, d, e, f\}$ for the degree 0 and 1 polynomials on N_u. With respect the former linearly independent set and the second basis, the map $p \in P^3_H(U)$ to $\phi^*_u(p)$, the restriction to the slice, followed by truncation to degree 1, is represented by the matrix

$$A = \begin{bmatrix} 1 & \lambda^2 & \mu^2 & \lambda\mu\nu & 0 & 0 & 0 \\ 0 & 2\lambda & 0 & \mu\nu & 0 & 0 & 0 \\ 0 & 0 & 2\mu & \lambda\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\lambda^2+\mu^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda^2+\nu^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\mu^2+\nu^2} \end{bmatrix}.$$

This matrix has nonzero determinant since $\lambda \geq \mu > 0$. This proves that $P_H^3(U)$ is first order abundant at u, and hence $P_H^4(U)$ is, also.

Part (ii). The Decomposable Case ($H_u \cong U(1)$).
Assume that $u = (0, v) \in \underline{\mathbb{R}}^k \oplus (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^k)$ where $v = (\lambda e_1, 0, \ldots, 0)$. The orbit of u is 2-dimensional, and $T_u\mathcal{O}_u$ is spanned by $(0, (e_2, 0, \ldots, 0))$ and $(0, (e_3, 0, \ldots, 0))$. It will be helpful to simplify the notation if we identify $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$, since $H_u \cong U(1)$ acts on $\mathbb{R}^3$ in this fashion. For a complex coefficient $z = a + ib$, let ze_2 denote $ae_2 + be_3$. In this way, we obtain a coordinate system $(x_1, \ldots, x_k, (y_1, z_1), \ldots, (y_l, z_l))$ on U. The normal space N_u to the orbit is parameterized by the real variables $x_1, \ldots, x_k, y_1, \ldots, y_l$ and the complex variables $z_2, \ldots, z_l$.

The slice parameterization at u is

$$\phi_u(x_1, \ldots, x_k, y_1, \ldots, y_l, z_2, \ldots z_l) = \\ (x_1, \ldots, x_k, ((\lambda + y_1)e_1, y_2 e_1 + z_2 e_2, \ldots, y_l e_1 + z_l e_2)).$$

This time, we'll write out the proof in the case when $k = 1, l = 3$. Consider the following collection of invariant polynomials

$$\Big\{1, x_1, x_1^2, \|v_1\|^2, \|v_1\|^4, x_1\|v_1\|^2, \langle v_1, v_2\rangle, \ \|v_1\|^2\langle v_1, v_2\rangle, x_1\langle v_1, v_2\rangle, \\ \|v_1 \wedge v_2\|^2, \|v_2\|^2, \langle v_1, v_3\rangle, \|v_1\|^2\langle v_1, v_3\rangle, x_1\langle v_1, v_3\rangle, \|v_1 \wedge v_3\|^2, \|v_3\|^2, \\ \langle v_1, v_2\rangle\langle v_1, v_3\rangle, \langle v_2, v_2\rangle, \det[v_1 v_2 v_3]\Big\}$$

and the basis

$$\Big\{1, x_1, x_1^2, y_1, y_1^2, x_1 y_1, \ y_2, y_1 y_2, x_1 y_2, |z_2|^2, y_2^2,$$

$$y_3, y_1y_3, x_1y_3, |z_3|^2, y_3^2\ y_2y_3, \langle z_2, z_3\rangle, \langle z_2, iz_3\rangle\} \subset P_H^2(N_u).$$

Then the map which sends a polynomial $p \in P_H^4(U)$ to $\phi_u^*(p)$, truncated to degree 2, is represented by the matrix

$$\begin{bmatrix}
1&0&0&\lambda^2&\lambda^4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&\lambda^2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2\lambda&4\lambda^3&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&6\lambda^2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2\lambda&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\lambda&\lambda^3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&3\lambda^2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\lambda&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\lambda^2&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\lambda&\lambda^3&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&3\lambda^2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\lambda&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\lambda^2&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\lambda^2&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}.$$

This nearly diagonal matrix has nonzero determinant whenever $\lambda > 0$ is sufficiently small. $\square$

Proof of Theorem 53. We again demonstrate the appropriate order of abundance at the u in the canonical form above, for both the decomposable and indecomposable cases. In both these cases, the proof depends on certain bounds derived from Taylor's theorem. Let $f \in C_H^k(U)$ be a function which vanishes to order $k-1$ at 0, and let M denote an upper bound for $d^k f$ on some ball B about 0 containing u. Then the following bounds are straightforward applications of Taylor's Theorem. In the notation, t_i are all numbers between 0 and 1.

Taylor expansion bounds for the indecomposable case.
We'll write out the Taylor expansion inequalities for the case $u = (\lambda e_1, \mu e_2, \nu e_3) \in (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^3)$, leaving the general case of an indecomposable element of $\mathbb{R}^k \oplus (\mathbb{R}^3 \otimes \underline{R}^l)$ to the reader.

$$\begin{aligned} |f(u)| &= |d^k f_{t_0 u}(u, \ldots, u)| \\ &\leq M|u|^k \\ &\leq 3^{\frac{k}{2}} M \lambda^k. \end{aligned} \tag{11}$$

Furthermore,

$$\begin{aligned} df_u(e_1, 0, 0) &= d^2 f_{t_1 u}((e_1, 0, 0), u) \\ &= \cdots \\ &= d^k f_{t_{k-1} u}((e_1, 0, 0), u, t_1 u, t_{k-2} u), \end{aligned}$$

and so

$$|df_u(e_1, 0, 0)| \leq 3^{\frac{k-1}{2}} M \lambda^k. \tag{12}$$

Finally, for

$$v \perp \left(e_1 \otimes \mathbb{R}^l\right) = span\{(e_1, 0, \ldots, 0), (0, e_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, e_1)\},$$

we get the following improved bound, which stems from the fact that H invariance implies $df_{(\lambda e_1, 0, 0)}(v) = 0$.

$$\begin{aligned} & df_u(v) \\ &= d^2 f_{(\lambda e_1, t_1 \mu e_2, t_1 \nu e_2)}(v, (0, \mu e_2, \nu e_3)) \\ &= d^2 f_{(\lambda e_1, 0, 0)}(v, (0, \mu e_2, \nu e_3)) \\ &\quad + d^3 f_{(\lambda e_1, t_2 \mu e_2, t_2 \nu e_3)}(v, (0, \mu e_2, \nu e_3), (0, t_1 \mu e_2, t_1 \nu e_3)) \\ &= d^3 f_{(t_3 \lambda e_1, 0, 0)}(v, (0, \mu e_2, \nu e_3), (\lambda e_1, 0, 0)) \\ &\quad + d^4 f_{(\lambda e_1, t_4 \mu e_2, t_4 \nu e_3)}(v, (0, \mu e_2, \nu e_3), (0, t_1 \mu e_2, t_1 \nu e_3), (0, t_2 \mu e_2, t_2 \nu e_3)) \\ &= \cdots \\ &= d^k f_{(t_{2k-1} \lambda e_1, 0, 0)}(v, (0, \mu e_2, \nu e_3), (\lambda e_1, 0, 0), (t_3 \lambda e_1, 0, 0), \ldots, (t_{2k-3} \lambda e_1, 0, 0)) \\ &\quad + d^k f_{(\lambda e_1, t_{2k-4} \mu e_2, t_{2k-4} \nu e_3)}(v, (0, \mu e_2, \nu e_3), (0, t_1 \mu e_2, t_1 \nu e_3), \\ &\qquad\qquad (0, t_2 \mu e_2, t_2 \nu e_3), \ldots, (0, t_{2k-6} \mu e_2, t_{2k-6} \nu e_3)). \end{aligned}$$

The last equation gives the following bound (for $v \perp \left(e_1 \otimes \mathbb{R}^l\right)$)

$$
\begin{aligned}
|df_u(v)| &\leq M(\lambda^{k-2}(\mu^2+\nu^2)^{\frac{1}{2}} + (\mu^2+\nu^2)^{\frac{k-1}{2}})\|v\| \\
&\leq 2^{\frac{k+1}{2}} M \lambda^{k-2} \mu \|v\|. \qquad (13)
\end{aligned}
$$

Taylor expansion bounds for the decomposable case.
Again, we assume that $u = (0, (\lambda e_1, 0, \ldots, 0)) \in \underline{\mathbb{R}}^k \oplus (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l)$. In this case, one immediately gets the bounds

$$|f(u)| \leq M\lambda^k \qquad (14)$$

$$|df_u(v)| \leq M\lambda^{k-1}\|v\| \qquad (15)$$

$$|d^2 f_u(v,w)| \leq M\lambda^{k-2}\|v\|\|w\|. \qquad (16)$$

We now explain how to modify the proof of Lemma 50 to deal with the present cases. Assume that $V \subset C^6_H(U)$ is fourth order abundant at $0 \in U$. We first prove that, for $\lambda > 0$ sufficiently small and $\lambda \geq \mu > 0$ with $\mu \geq \nu \geq 0$, fourth order abundance at 0 implies that V is first order abundant at $u = (\lambda e_1, \mu e_2, \nu e_3) \in \mathbb{R}^3 \otimes \underline{\mathbb{R}}^3$. (The cases $l = 2$ or $l > 3$ require only straightforward modifications.)

Choose a collection $f_1, \ldots, f_m \in V$ such that the 4-jets $j^4_0(f_i)$ agree with those of the H invariant polynomials listed in equation (10). Let M be an upper bound for all the fifth derivatives $d^5 f_i)$ on some neighborhood $B_U \subset U$ of 0.

The Taylor expansions of these functions (more precisely, $f_i \circ \phi_u$) in the variables $(a, \ldots, f)$ are given by a matrix of the form $\widetilde{A} = A + \Delta$ where the columns of Δ are the Taylor expansions (about u) of the errors (the differences between each f_i and its fourth order Taylor expansion about 0). Inequality (11) implies that all entries in the first row of Δ are bounded by $M'\lambda^5$. Inequality (12) implies that the entries in the second, fifth and sixth rows are bounded by $M'\lambda^4$, and the inequality (13) implies that the entries of the third, fourth and seventh rows are bounded by $M'\lambda^3\mu$.

It follows that $\det \widetilde{A}$ can be expressed as $\lambda^4\mu^3$ multiplied by the determinant of another matrix (obtained from $\widetilde{A}$ by dividing rows 2, 3, 4, 5, 6, 7 by λ, μ,

$\lambda\mu$, λ, λ and μ, respectively. This new matrix has the form

$$\begin{bmatrix} 1 & * & * & * & * & * & * \\ * & 2 & * & * & * & * & * \\ * & * & 2 & * & * & * & * \\ * & * & * & 1 & * & * & * \\ * & * & * & * & \delta_1 & * & * \\ * & * & * & * & * & \delta_2 & * \\ * & * & * & * & * & * & \delta_3 \end{bmatrix},$$

where $\delta_1 = (1 + (\frac{\mu}{\lambda})^2)^{\frac{1}{2}}$, $\delta_2 = (1 + (\frac{\nu}{\lambda})^2)^{\frac{1}{2}}$, and $\delta_3 = (1 + (\frac{\mu}{\nu})^2)^{\frac{1}{2}}$, and where all asterisks indicate entries that are bounded by $M'\lambda^2$. Since all the diagonal entries are greater than one, it is therefore clear that for $\lambda > 0$ sufficiently small, $\widetilde{A}$ has full rank. This completes the proof of first order abundance at $u = (\lambda e_1, \mu e_2, \nu e_3)$, for $\lambda \geq \mu > 0$, $\mu \geq \nu \geq 0$, when λ is sufficiently small.

The proof in the decomposable case (in canonical form) is similar but easier, using the inequalities (14), (15), and (16). The rest of the argument in Subsection 5.1 works the same for the present Lie group H and representation U. □

5.3 $U = \underline{\mathbb{R}}^k \oplus (\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l)$ *and* $H = U(2)$.

In this subsection we prove an analog of Theorem 53 for this group and representation. Here, H acts by the standard action on $\mathbb{C}^2$. The proof is quite similar to the preivious case, so we will only briefly outline it.

Theorem 56 *Theorem 53 applies to* $H = U(2)$ *and* $U \cong \underline{\mathbb{R}}^k \oplus (\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l)$ *as well.*

Outline of proof. The proof of abundance of the space of invariant polynomials is nearly identical, but of course the polynomials are slightly different. Up to the action of $K' = U(2) \times SO(l)$, any nonzero $u \in U_H^\perp$ can be assumed to have the form $(0, (v_1, \ldots, v_l))$ where $v_1 = \lambda e_1$, $v_2 = \mu e_2$, and $v_3 = \cdots = v_l = 0$ for real numbers $\lambda \geq \mu \geq 0$ and $\{e_1, e_2\}$ is the standard complex basis for $\mathbb{C}^2$. Here, μ is zero or nonzero, depending on whether u is decomposable or indecomposable.

In the invariant polynomials on U, there is no determinant $\det[v_1 v_2 v_j]$ as in the previous section, but there are polynomials $\langle v_i, i v_j \rangle$. (The inner product here denotes the real inner product, so this quantity equals minus the imaginary part of the Hermitian inner product on $\mathbb{C}^2$ evaluated on v_i and v_j). The Taylor expansion bounds needed in the present case are completely parallel to those used in the last section. $\Box$

5.4 $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2} \oplus \mathbb{C}^{l_3}$ *and* $H = U(1) \times U(1) \times U(1)$.

Consider the group $H = U(1) \times U(1) \times U(1)$, acting on $\underline{\mathbb{R}}^k \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2} \oplus \mathbb{C}^{l_3}$ by the action

$$(\lambda, \lambda', \lambda'')(x, z, z', z'') = (x, \lambda'(\lambda'')^{-1}z, \lambda(\lambda'')^{-1}z', \lambda(\lambda')^{-1}z'').$$

Note that this action is how the scalar $U(3)$ elements act on the off-diagonal elements of a $u(3)$ matrix under the adjoint action.

In this H vector space the top stratum consists of points $u = (x, z, z', z'')$ where at least 2 of the components z, z', and z'' are nonzero. These points have stabilizer $\{(\lambda, \lambda, \lambda) \mid \lambda \in U(1)\}$ (such group elements act trivially on all of U). The fixed point stratum consists of points $u = (x, 0, 0)$, with $H_u = H$. There are also three intermediate strata, consisting of points where exactly one of the components z, z' and z'' is nonzero. In either case, H_u is a subgroup of H isomorphic to $U(1) \times U(1)$.

Theorem 57 *The conclusion of Theorem 53 also applies to the group* $H = U(1) \times U(1) \times U(1)$ *and the* H *vector space* $U = \underline{\mathbb{R}}^k \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2} \oplus \mathbb{C}^{l_3}$*, with the action specified above.*

PROOF. The proof of involves calculations similar to those in Subsection 5.2, but with the invariant polynomials of degree ≤ 4 on this H vector space. These are no harder to understand than those in Subsection 5.1, although they are more plentiful. We leave it as an exercise for the interested reader. $\Box$

6 Gauge theory on 3-manifolds

In this section we discuss holonomy perturbations of the flatness equation for connections on a 3-manifold. We begin by recalling the the basic analytical framework of [20,**?**,**?**]. Then we describe the space of admissible holonomy perturbations. This space of perturbations is fairly standard, except that we require the perturbation curves to share a common basepoint, due to the more subtle transversality requirements (higher order abundance) for the multiple strata of reducible flat connections.

In Subsection 6.3, we recall the Fredholm equation which is (locally) equivalent to the flatness equation on connections modulo gauge and show that the discussion of perturbations of Fredholm exact 1-forms applies in this setting. We demonstrate abundance properties of the space of admissible perturbations at the flat connections in Subsection 6.4. The extension results of Section 5 are required to obtain the necessary neighborhood of the flat moduli space where abundance holds.

Global transversality conclusions for the $SU(3)$, $U(3)$, $SU(4)$ and $U(4)$ moduli spaces analogous to Theorems 43 and 46 are stated in Subsection 6.5.

6.1 *Sobolev completions of the spaces of connections and gauge transformations*

Let X^3 be a 3-dimensional manifold. Fix a compact Lie group G, and a principal bundle $G \to P \to X$. We'll assume that $G \subset U(n)$, so that $-tr(\cdot,\cdot) : Lie(G) \times Lie(G) \to \mathbb{R}$ defines an adjoint invariant inner product on the Lie algebra. Let $adP = P \times_{ad} Lie(G)$ denote the associated adjoint bundle. Given any smooth connection Γ on P, any other connection on P differs by a differential 1-form on X with values in adP. We will let $\Omega^p(X; adP)$ denote the space of differential p-forms with values in adP, in other words, the space of sections of $\Lambda^p(T^*X) \otimes adP$.

Let $\mathcal{A}$ denote the space of L^2_1 connections on P, i.e. connections that differ from a smooth connection by an element of $L^2_1\Omega^1(X; adP)$. Let $\mathcal{G}$ denote the set of L^2_2 gauge transformations, i.e. G equivariant bundle isomorphisms $g : P \to P$ covering the identity map $id : X \to X$. The gauge group $\mathcal{G}$ acts on $\mathcal{A}$, according to the formula

$$g \cdot A = gAg^{-1} - (d_A g)g^{-1}.$$

The action of $\mathcal{G}$ on $\mathcal{A}$ is smooth, and the quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ is a Hausdorff space. See [7] and [8] for details.

Specify a Riemannian metric on X. This determines a Hodge star operator $* : \Omega^p(X; adP) \to \Omega^{3-p}(X; adP)$. Combining this with the inner product on the fiber $Lie(G)$ of $ad(P)$, we obtain an L^2 inner product on forms

$$\langle \omega_1, \omega_2 \rangle_{L^2} = \int_X -tr(\omega_1 \wedge *\omega_2).$$

The gauge group $\mathcal{G}$ forms a Hilbert group, with Lie algebra $L^2_2\Omega^0(X; adP)$.

The linearization of the group action at A may be identified with the map

$$-d_A : L^2_2\Omega^0(X; adP) \to L^2_1\Omega^1(X; adP).$$

With respect to the L^2 inner product on $T_A\mathcal{A} = L^2_1\Omega^1(X; adP)$, the orthogonal complement to the orbit tangent space $T_A\mathcal{O}_A$ is $\ker d^*_A \cap L^2_1\Omega^1(X; adP)$, where

$$d^*_A : L^2_1\Omega^1(X; adP) \to L^2\Omega^0(X; adP)$$

is the adjoint of d_A.

A slice for the gauge group action through A is given by neighborhood of A in $A+L^2_1\mathcal{X}_A$, where $\mathcal{X}_A = \ker d^*_A \cap \Omega^1(X; adP)$. The stabilizer $stab(A)$ of the connection A is a subgroup of the gauge group. For any $x_0 \in X$, the map sending each $g \in stab(A)$ to $g(x_0) \in (P \times_{Ad} G)|_{x_0} \cong G$ is an injective homomorphism and we can view $stab(A)$ as a subgroup of G (it is the centralizer of the holonomy subgroup based at x_0). The standard slice theorem in gauge theory shows that, for sufficiently small $stab(A)$ invariant neighborhoods $B \subset A + L^2_1\mathcal{X}_A$ of A, there is an embedding

$$B \times_{stab(A)} \mathcal{G} \to \mathcal{A}.$$

Thus, the gauge orbit of $[A]$ has a neighborhood in $\mathcal{A}/\mathcal{G}$ modeled on $B/stab(A)$.

Associated to each connection $A \in \mathcal{A}$ is its curvature 2-form $F(A) \in L^2\Omega^2(X; adP)$. If A_0 is a smooth connection, then $F(A_0)$ is a smooth 2-form, and if $A = A_0+a$ with $a \in L^2_1\Omega^1(X; adP)$, then $F(A) = F(A_0) + d_{A_0}a + a \wedge a$, which lies in L^2 by standard elliptic theory and the Sobolev Multiplication Theorem $L^2_1 \times L^2_1 \to L^2$. Here $a \wedge a = \frac{1}{2}[a \wedge a]$ is an adP valued 2-form.

A connection is said to be flat if its curvature is zero. This is a gauge invariant condition, since $F(gA) = gF(A)g^{-1}$. The flat moduli space is defined to be the quotient space,

$$\mathcal{M} = \{A \in \mathcal{A} \mid F(A) = 0\}/\mathcal{G}.$$

In the next subsection we will demonstrate that the flatness equation is locally equivalent to a Fredholm equation.

We will interpret the flat moduli space as the set of critical points of the Chern-Simons function, defined as follows. Let A_0 be a smooth connection. Then, if $A = A_0 + a$,

$$\mathrm{cs}_{A_0}(A) = \frac{1}{8\pi^2}\int_X tr\left(2a \wedge F(A_0) + a \wedge d_{A_0}a + \tfrac{2}{3}a \wedge a \wedge a\right).$$

Since the choice of A_0 only affects the Chern-Simons function through an additive constant, we will typically drop the A_0 subscript.

Given the inner product pairing $-tr$ on $Lie(G)$, we obtain a nondegenerate bilinear pairing between smooth adP valued 1-forms and adP valued 2-forms by sending

$$(\omega_1, \omega_2) \in \Omega^1(X; adP) \times \Omega^2(X; adP) \mapsto -\int_X tr(\omega_1 \wedge \omega_2) \in \mathbb{R}.$$

After completing with Sobolev norms, this identifies $L^2_{-1}\Omega^2(X; adP)$ with the dual space of $L^2_1\Omega^1(X; adP)$. Under this identification, the differential of the Chern-Simons function is given by $d\,\mathrm{cs}_A = -\frac{1}{4\pi^2}F(A)$.

We will discuss the Fredholm properties of the flatness equation on the slice in Subsection 6.3, after we define the space of admissible functions which will be used to perturb the Chern-Simons function.

6.2 Admissible functions

In this subsection, we will recall the definition of holonomy perturbations and discuss their properties. We begin by fixing a collection of closed embedded loops $l_1, \ldots, l_N$ in X, disjoint except for a common basepoint x_0, such that the homotopy classes $[l_i]$ generate $\pi_1 X$. We also require that the tangent directions agree at the basepoint. Let $\gamma_i : S^1 \times D^2 \to X$ be diffeomorphisms from the solid torus to tubular neighborhoods of the curves l_i. These are to be chosen so that $\gamma_i(1, x) = \gamma_j(1, x)$, i.e. there is a common normal disk at the basepoint.

Fix a radially symmetric bump function $\beta : D^2 \to \mathbb{R}$ satisfying $\beta(x) \geq 0$ for all x, $\beta(x) = 0$ when $\|x\| = 1$, and $\int_{D^2} \beta(x)dx = 1$.

Suppose that $\overline{f} : G^N \to \mathbb{R}$ is a C^s invariant function which is AdG invariant, i.e.

$$\overline{f}(g^{-1}g_1g, g^{-1}g_2g, \ldots, g^{-1}g_Ng) = \overline{f}(g_1, \ldots, g_N).$$

We will form a C^s function $f : \mathcal{A} \to \mathbb{R}$ by

$$f(A) = \int_{D^2} \overline{f}(hol_{\gamma_1(x)}(A), \ldots, hol_{\gamma_N(x)}(A))\beta(x)dx. \tag{17}$$

For a smooth connection A, the holonomy $hol_{\gamma_i(x)}(A)$ around $\gamma_i(S^1 \times \{x\})$ depends continuously on $x \in D^2$, so this makes sense at least for smooth connections.

Lemma 58 *The function f on the space of smooth connections defined above extends to a C^s function on the L^2_1 Sobolev completion $\mathcal{A}$.*

PROOF. Here we follow Section 8a of [20]. The Lie group G, by assumption, is a submanifold of the vector space $End(\mathbb{C}^n)$. If $\overline{f} : G^N \to \mathbb{R}$ is a C^s function, then by choosing a tubular neighborhood of $G \subset End(\mathbb{C}^n)$ it is easy to give an extension of $\overline{f}$ to all of $End(\mathbb{C}^n)^N$ in such a way that the extension (which we also denote by $\overline{f}$) still is C^s bounded. The advantage of doing so is that, for a path $\alpha(t) = (g_1(t), \ldots, g_N(t)) \in G^N$, we can now differentiate $\overline{f}(\alpha(t))$ multiple times using the chain rule. Namely, we obtain a formula for $\frac{d^k}{dt^k}\overline{f}(\alpha(t))$ which is a linear combination of terms of the form

$$d^l\overline{f}_{\alpha(0)}(\alpha^{(r_1)}(0), \ldots, \alpha^{(r_l)}(0)), \tag{18}$$

where $\alpha^{(r_i)}(0)$ denotes the r_i-th derivative, and $r_1 + \ldots + r_l = k$.

Consider for the moment the circle S^1 and the principal bundle $P = S^1 \times G$ over it. Let A be a connection on P. If $p : [0,1] \to S^1$ is a parameterization, we can trivialize $p^*(P)$ using the connection $p^*(A)$. The holonomy $hol(A)$ is the automorphism by which the fibers $p^*(P)|_1$ is identified with $p^*(P)|_0$ under the map from $p^*(P)$ to P.

Let $a \in \Omega^1(S^1; adP)$ and set $\tilde{a} = p^*a$, which we interpret as a map $\tilde{a} : [0,1] \to Lie(G)$ using the trivialization. Let $P(s;t)$ denote parallel translation from 0 to s using the connection $p^*(A + ta)$. Then $P(s;t)$ solves the differential equation

$$\tfrac{\partial}{\partial s}P(s;t) + t\tilde{a}(s)P(s;t) = 0.$$

The holonomy $hol(A + ta)$ is given by $hol(A)P(1;t)$.

Differentiation of both sides of the differential equation with respect to t gives, for each k,

$$\tfrac{\partial^k}{\partial t^k}\tfrac{\partial}{\partial s}P(s;t) + k\tilde{a}(s)\tfrac{\partial^{k-1}}{\partial t^{k-1}}P(s,t) + t\tilde{a}(s)\tfrac{\partial^k}{\partial t^k}P(s;t) = 0.$$

Using this formula for $t = 0$ inductively one can show that

$$\tfrac{d^k}{dt^k}hol(A+ta)|_{t=0} = \pm hol(A)\int_0^1 \tilde{a}(s_1)\int_0^{s_1}\tilde{a}(s_2)\int_0^{s_2}\cdots\int_0^{s_{k-1}}\tilde{a}(s_k)ds_k\cdots ds_1.$$

This gives a bound

$$\left|\tfrac{d^k}{dt^k}hol(A+ta)|_{t=0}\right| \le C|a|^k_{L^1(S^1)}.$$

Returning to the admissible functions defined above, we can apply inequalities like this for each of the N holonomies (for fixed x) to each of the terms of the form (18), obtaining a bound similar to equation (8.4) in [20], namely

$$\left|\tfrac{d^k}{dt^k} f(A+ta)\right| \leq \text{const}\left|d^k \overline{f}\right| \sum_{1\leq i_1,\ldots,i_k\leq N} \int_{D^2} \left(|a|_{L^1(\gamma_{i_1}(x))} \cdots |a|_{L^1(\gamma_{i_k}(x))}\right) \beta(x)dx.$$

The rest of the argument now follows as in [20]. $\square$

Proposition 59 *For any L^2_1 connection A and any admissible function f, the differential df_A is an element of L^2.*

PROOF. The bound above, in the case $k = 1$, shows immediately that

$$\|df_A(a)\|_{L^2(X)} \leq K\|\overline{f}\|_{C^s(G^N)}\|a\|_{L^1(X)} \leq C\|a\|_{L^2_1(X)}.$$

This shows that df_A is in the dual space L^2_{-1}. In fact, since $\|a\|_{L^1} \leq C\|a\|_{L^2}$ on a compact manifold, the differential of f at A is bounded on L^2, which means that $df_A \in L^2\Omega^1(X; adP)$, the same place that $d\,\mathrm{cs}_A = -\frac{1}{4\pi^2}F(A)$ resides. $\square$

Let $\mathcal{F} = C^s_{AdG}(G^N)$. We interpret elements of $\mathcal{F}$ as functions on $\mathcal{A}$, via the map $\overline{f} \mapsto f$ described above. It is clear from the construction that the resulting functions are gauge invariant, so the map goes from $\mathcal{F}$ to $C^s_{\mathcal{G}}(\mathcal{A})$.

6.3 The local Fredholm equation for the perturbed flat moduli space

We now outline the local Fredholm model for the flat moduli space. Let A be a flat connection; up to gauge transformation, it may be assumed to be smooth. As in Section 6.1, we set $L^2_1\mathcal{X}_A = \ker d^*_A \cap L^2_1\Omega^1(X; adP)$ equal to the slice tangent space. Consider the map $F : L^2_1\mathcal{X}_A \to L^2\Omega^2(X; adP)$ sending $a \mapsto F(A + a)$. This linearization is $d_A|_{L^2_1\mathcal{X}_A} : L^2_1\mathcal{X}_A \to L^2\Omega^2(X; adP)$, which has finite dimensional kernel (equal to $H^1_A(X; adP)$) but has infinite dimensional cokernel (including all of $d^*_A(L^2_1\Omega^3(X; adP))$). To replace this map with a Fredholm one, we set $L^2(*\mathcal{X}_A) = \ker d_A \cap L^2\Omega^2(X; adP)$ and let $\Pi_A : L^2\Omega^2(X; adP) \to L^2(*\mathcal{X}_A)$ denote the L^2 orthogonal projection onto this subspace.

Lemma 60 *For $\|a\|_{L^2_1}$ sufficiently small, the restriction of the projection map Π_A to $\ker d_{A+a} \cap L^2\Omega^2(X; adP)$ is injective.*

PROOF. This is Lemma 12.1.2 in [17]. $\square$

Proposition 61 *There is a neighborhood $B \subset L^2_1\mathcal{X}_A$ of zero such that, for any $f \in \mathcal{F}$ and $a \in B$, the equation $d(\mathrm{cs}+f)_{A+a} = 0$ is equivalent to $\Pi_A(d(\mathrm{cs}+f)_{A+a}) = 0$.*

PROOF. This follows from the fact that f is gauge invariant and cs is invariant under the identity component of $\mathcal{G}$ (the latter claim is equivalent to the Bianchi identity). Let $A' = A + a$ be a connection. We claim that $d(\mathrm{cs}+f)_{A'} \in \ker d_{A'} \cap L^2\Omega^2(X; adP)$. Given this fact, the proposition follows from Lemma 60.

To prove the claim, we will use the orthogonal decomposition

$$L^2\Omega^2(X; adP) = \Big(\ker d_{A'} \cap L^2\Omega^2(X; adP)\Big) \oplus d^*_{A'} L^2_1\Omega^3(X; adP).$$

We will show that $d(\mathrm{cs}+f)_{A'}$ is orthogonal to $d^*_{A'}\left(L^2_1\Omega^0(X; adP)\right) = *d_{A'}L^2_1\Omega^0(X; adP)$. Choose $\eta \in L^2_1\Omega^0(X; adP)$. Choose a sequence of L^2_2 0-forms η_i which limit to η in the L^2_1 norm. Then $*d_{A'}\eta_i$ is a sequence of tangent vectors to the orbit $\mathcal{O}_{A'}$ which converge to $*d_{A'}\eta$ in L^2. Hence

$$\begin{aligned}
\langle *d_{A'}\eta, d(\mathrm{cs}+f)_{A'}\rangle_{L^2} &= \lim_{i\to\infty} \langle *d_A\eta_i, d(\mathrm{cs}+f)_{A'}\rangle_{L^2} \\
&= \lim_{i\to\infty} d(\mathrm{cs}+f)_{A'}(d_A\eta_i) \\
&= \lim_{i\to\infty} \tfrac{d}{dt}(\mathrm{cs}+f)\left(\exp(t\eta_i)\cdot A'\right)|_{t=0} \\
&= 0.
\end{aligned}$$

□

We complete the subsection by noting that the Fredholm map

$$a \in B \subset L^2_1\mathcal{X}_A \mapsto \Pi_A\left(-\tfrac{1}{4\pi^2}F(A+a) + df_{A+a}\right) \in \ker d_A \cap L^2\Omega^2(X; adP)$$

is the differential of the function on B defined by $a \mapsto (\mathrm{cs}+f)(A+a)$, i.e. the restriction of $\mathrm{cs}+f$ to the slice through A. It follows that the general comments concerning Fredholm exact 1-forms in Section 2.2 apply. We will address the issue of abundance at the flat connections in the next subsection. Abundance nearby will be obtained through the extension results in Section 5. Before we get into the abundance questions, we pause a moment to relate the gauge theory notation with the notation used in the general discussion.

Let A be a smooth flat connection. For the two Banach spaces E and $F \subset E^*$, we take $E = L^2_1\mathcal{X}_A = \ker d^*_A \cap L^2_1\Omega^1(X; adP)$ and $F = \ker d_A \cap L^2\Omega^2(X; adP) = L^2(*\mathcal{X}_A)$. For the Fredholm map $\zeta : E \to F$, we take

$$\zeta(a) = \Pi_A(d(\mathrm{cs}+f)_{A+a}) = \Pi_A\left(-\tfrac{1}{4\pi^2}F(A+a) + df_{A+a}\right).$$

The kernel $U = \ker d\zeta_0$ is equal to the first cohomology of X with coefficients in the flat $Lie(G)$ bundle associated to A, $U = H^1_A(X; adP)$. (More specifically, U is the space of harmonic 1-forms with respect to d_A.) The image $V' = \mathrm{Image}(d\zeta_0)$ in the present context is

$$V' = \mathrm{Image}\left(d_A : \ker d_A^* \cap L^2_1\Omega^1(X; adP) \to \ker d_A \cap L^2\Omega^2(X; adP)\right).$$

We take as V and W the L^2 orthogonal complements of U and V' in E and F, respectively. (Note that, by the Hodge theorem, $W = H^2_A(X; adP)$ is the dual space to U and annihilates V.) Thus Φ, in this context, is a map from $H^1_A(X; adP)$ to $H^2_A(X; adP) = *H^1_A(X; adP)$.

In the gauge theory setting, the map ζ is equivariant with respect to the compact Lie group $stab(A)$, so this plays the role of the group H in Section 4.

6.4 Abundance at the flat connections

Let A be a smooth flat connection, and let U, V, W, V' be as defined at the end of the last subsection. Consider $\mathcal{F} = C^s_{AdG}(G^N)$. Choose neighborhoods B_U of $0 \in U$ and $B_{\mathcal{F}}$ of $0 \in \mathcal{F}$, with B_U invariant under the $H = stab(A)$ action, and sufficiently small that the parameterized map $\psi : B_U \times B_{\mathcal{F}} \to V$ is well-defined, and define Ψ and Φ accordingly as in Section 2.3.

Let $\mathcal{T}^A : B_{\mathcal{F}} \to C^{s-1}_H(B_U)$ be the map

$$\mathcal{T}^A(f)(a) = (\mathrm{cs} + f)(A + a + \psi(a, f)).$$

Establishing the abundance properties necessary to obtain transversality results for generic (small) perturbations and paths of perturbations will be a several step process. We begin the process by establishing abundance at $(0, 0) \in U \times \mathcal{F}$.

Proposition 62 *If $s > k + 1$, then the map $\mathcal{T}^A : B_{\mathcal{F}} \to C^{s-1}_H(B_U)$ is kth order abundant at $(0, 0)$.*

Remark 63 *This proposition in the $SU(3)$ case with $k = 2$ corrects Proposition 3.4 (i) (a) of [3].*

To outline the proof of this proposition, we need some additional notation. Let $\gamma_1, \ldots, \gamma_N$ be the collection of thickened curves used to define the perturbations. We'll adopt the abbreviated notation

$$\overline{hol}_x(A) = (hol_{\gamma_1(S^1 \times \{x\})}(A), \ldots, hol_{\gamma_N(S^1 \times \{x\})}(A)) \in G^N.$$

We have an integration map (also dependent on A, but we leave this out of the notation)

$$I : D^2 \to \left(adP|_{\gamma_i(\{1\}\times D^2)} \right)^N ,$$

defined as follows.

Fix a parameterization (which we also denote by γ_i) for each solid torus, $\gamma_i : [0,1] \times D^2 \to X$ which descends to an embedding of the solid torus when the disks at the ends are identified. Given $x \in D^2$ and $a \in \Omega^1(X; adP)$, we define

$$I_x(a) = \left(\int_0^1 P_{1,x}(s)^*(a(\gamma_{1_*}(\tfrac{\partial}{\partial s})))ds, \ldots, \int_0^1 P_{N,x}(s)^*(a(\gamma_{N_*}(\tfrac{\partial}{\partial s})))ds \right),$$

where $P_{i,x}$ denotes parallel translation along $\gamma_i(S^1 \times \{x\})$ from $\gamma_i(0,x)$ to $\gamma_i(s,x)$ using the connection A. Here $\gamma_{i*}(\frac{\partial}{\partial s})$ denotes the velocity vector of the parameterization γ_i with respect to the parameter s, in other words $\frac{\partial}{\partial s}\gamma_i(s,x)$. The upper asterisk denotes the pull-back, i.e.

$$P_{i,x}(s)^*(a(\gamma_{i_*}(\tfrac{\partial}{\partial s}))) = P_{i,x}(s)^{-1} \circ a(\gamma_{i_*}\tfrac{\partial}{\partial s})) \circ P_{i,x}(s).$$

Pulling back in this way gives a map from the interval to the fiber $adP|_{\gamma_i(0,x)}$, which can then be integrated.

All calculations concerning the holonomies around $\gamma_i(S^1 \times \{x\})$ can be performed by pulling connections on (the bundle over) the solid torus back to connections on $[0,1] \times D^2$. In particular, to differentiate $hol_{\gamma_i(S^1\times x)}(A)$ with respect to A, we use parallel translation for $\gamma_i^*(A)$ to trivialize the pulled back adjoint bundle. One must keep in mind that the trivialization does not descend to a section of $adP|_{\gamma_i(S^1\times D^2)}$ because the bundle fibers at the ends are identified via the holonomy. One can then check that, for smooth connections A and smooth 1-forms $a \in \Omega^1(X; adP) = T_A\mathcal{A}$,

$$\frac{d}{dt}\overline{hol}_x(A+ta)|_{t=0} = \overline{hol}_x(A)I_x(a).$$

(The smoothness assumption is not a problem for us since we will only use this fact at flat connections, which are smooth up to gauge, and on harmonic 1-forms, which are also smooth by elliptic regularity.)

The proof of Proposition 62 will be based on the following lemma. For A flat, we can assume by gauge transforming that the holonomies of A are independent of x, and we obtain (via parallel translation over the basepoint disk) an identification of all the fibers of $adP_{\gamma_i(\{0\}\times D^2)}$, so we may view I as a map taking values in a single fiber. This is naturally identified with the tangent space to G at $hol_{\gamma_i(S^1\times\{x\})}(A)$.

Lemma 64 *Let A be a smooth flat connection, let $I : H^1_A(X; adP) \to T_{\overline{hol}(A)}G^N$ be the integration map described above, restricted to $H^1_A(X; adP)$. Let*

$$p : T_{\overline{hol}(A)}G^N \to T_{\overline{hol}(A)}G^N / T_{\overline{hol}(A)}[A]$$

denote the quotient map. Then $p \circ I_x$ is independent of x and $p \circ I$ is injective.

PROOF. The proof of the first claim is simply a calculation. The main tools are the following rules for integrating along a composition of paths (here integration means pulling back to the initial point via parallel translation and then integrating),

$$\int_{\beta\cdot\alpha} a = \int_\alpha a + (P_\alpha^{-1})^* \int_\beta a, \qquad \int_{\alpha^{-1}} a = -P_\alpha^* \int_\alpha a,$$

where P_α denotes parallel translation along α. Also note that tangent vectors to the orbit of $\overline{hol}(A)$ have the form

$$(hol_{\gamma_1}(A)m - mhol_{\gamma_1}(A), \ldots, hol_{\gamma_N}(A)m - mhol_{\gamma_N}(A))$$

for some $m \in Lie(G)$. We leave the details to the reader.

The proof of the second claim is fairly straightforward de Rham theory. Let W be the union of the thickened solid tori. Let $\rho = hol(A) : \pi_1(X) \to G$ denote the holonomy representation for A. We'll use the same name ρ for the representation on the free group generated by the fundamental group generators, $\rho : F_N = \pi_1(W) \to G$. The de Rham theorem identifies $H^1_A(X; adP) = H^1(\pi_1(X); ad_\rho)$ and $H^1_{A|_W}(W; adP|_W) = H^1(F_N; ad_\rho)$.

The standard complex for computing these group cohomologies gives a commutative diagram

$$\begin{array}{ccccc}
C^0(\pi_1(X); Lie(G)) & \xrightarrow{d^0} & C^1(\pi_1(X); Lie(G)) & \xrightarrow{d_R} & C^2(\pi_1(X); Lie(G)) \\
\cong\downarrow & & \cong\downarrow & & \downarrow \\
C^0(F_N; Lie(G)) & \xrightarrow{d^0} & C^1(F_N; Lie(G)) & \xrightarrow{0} & C^2(F_N; Lie(G)),
\end{array}$$

where d_R is a map obtained from differentiating the relations in this presentation of $\pi_1(X)$. In particular, C^0 and C^1 are identified, and hence the closed 1-cochains on $\pi_1(X)$ map injectively into the closed 1-cochains on F_N. $\square$

We now use the lemma to give a proof of the abundance property of the space of admissible perturbations at a flat connection.

Proof of Proposition 62. We begin by clarifying what must be proved. The final statement in Corollary 23 generalizes immediately to the equivariant setting. If the functions in $\mathcal{F}$ are invariant under a compact Lie group H action on $U \oplus V$ (which respects the direct sum decomposition), then the same statement holds with each S^l replaced by S^l_H. Thus, it is sufficient to show that for each $l \leq k$ there are admissible functions $f^l_1, \ldots, f^l_{m_l}$ which vanish to order $l-1$ at A and with the property that $d^l(f^l_j)_A$ restricted to $H^1_A(X; adP)$ (that is, pulled back to tensors on the cohomology via the inclusion of the cohomology into $T_A\mathcal{A}$) span the space of H invariant l tensors. We shall do this now.

Fix a flat connection A and an integer k. Gauge transform A if necessary to obtain a smooth gauge representative. Let $\overline{g} = \overline{hol}(A)$ and set $N_{\overline{g}} = T_{\overline{g}}G^N / T_{\overline{g}}\mathcal{O}_{\overline{g}}$. Set $H = G_{\overline{g}}$ equal to the stabilizer of $\overline{g}$ (i.e. the stabilizer of A in $\mathcal{G}$). Then H acts linearly on $N_{\overline{g}}$ and on $H^1_A(X; adP)$, and the map $p \circ I$ is equivariant. In particular, the image of $p \circ I$ is an invariant subspace, which we'll call N'. There is an equivariant isomorphism from $N_{\overline{g}}$ to $N' \oplus N''$ where N'' is some other representation of H.

As in the discussion surrounding Proposition 26, we let $i : N' \to N' \oplus N''$. Note that the pull back $i^* : S^l(N' \oplus N'') \to S^l(N')$ is surjective, and likewise $i^* : S^l_H(N' \oplus N'') \to S^l_H(N')$ is surjective. It is clear that for any $l \leq k$ we can find functions $\overline{f}^l_1, \ldots, \overline{f}^l_{m_l} \in C^k_H(N_{\overline{g}})$ which vanish to order $l-1$ at 0 and with the property that $d^l_0 \overline{f}^l_j, j = 1, \ldots, m_l$ span $S^l_H(N_{\overline{g}})$. In particular, we can choose them so that $i^* d^l_0 \overline{f}^l_j$ span $S^l_H(N')$.

Theorem 29 allows us to translate these functions (bumped off outside a neighborhood of the origin in $N_{\overline{g}}$) into a collection of G invariant C^k functions on G^N which vanish to order $l-1$ at $\overline{g}$ and with the corresponding spanning property for the l-th derivatives at $\overline{g}$.

We complete the proof by noting that if $\overline{f} \in C^k_G(G^N)$ vanishes to order $l-1$ at $\overline{g}$, then the corresponding admissible function $f : \mathcal{A} \to \mathbb{R}$ satisfies $d^j f_A = 0$ for $j = 0, \ldots, l-1$ and

$$\begin{aligned}
&d^l f(A)(a_1, \ldots, a_l) \\
&= \tfrac{d}{dt_1} \cdots \tfrac{d}{dt_l} \int_{D^2} \overline{f}\left(\overline{hol}_x(A + t_1 a_1 + \cdots + t_l a_l)\right) \beta x dx \Big|_{t_1 = \cdots = t_l = 0} \\
&= \int_{D^2} d^l \overline{f}_{\overline{g}} \left(\overline{hol}_x(A) I_x(a_1), \ldots, \overline{hol}_x(A) I_x(a_l)\right) \beta(x) dx.
\end{aligned}$$

For a_i closed, $\overline{hol}_x(A) I_x(a_i)$ is independent of x modulo tangent vectors to the orbit of $\overline{g}$. By Proposition 33, the quantity being integrated is independent of x, except for the bump function, which integrates to 1. Proposition 62 now follows. $\square$

6.5 Global transversality theorems

In this subsection, we state and prove global transversality results concerning the perturbed flat moduli space. Let X be a closed oriented 3-manifold. Recall that X is an integral homology sphere ($\mathbb{Z}$HS) if $H_*(X;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$, and X is a rational homology sphere ($\mathbb{Q}$HS) if $H_*(X;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. We will also assume that $G = SU(n)$ or $G = U(n)$, for $n \leq 4$.

We begin by noting some immediate consequence of the $\mathbb{Z}$HS or $\mathbb{Q}$HS assumptions on X, concerning orbit types of reducible flat connections and the cohomology at the reducible flat connections. We will again use the notation established in Section 1 for types of reducible connections.

We begin by recalling some well-known facts about $U(2)$ and $SU(2)$ connections. In general, $U(2)$ connections have type 2, $1+1$ or 1^2. These have stabilizer $\{\pm 1\}$, $U(1)$ and $SU(2)$ in $SU(2)$. In $U(2)$ the stabilizer is larger, containing the scalars, but these act trivially on $u(2)$ or $su(2)$ so, for the purpose of studying $stab(A)$ invariant functions on $H^1_A(X; adP)$, we can ignore this difference. Note also that for a $1+1$ (i.e. abelian noncentral) connection A, the action of the $U(1)$ stabilizer (in $SU(2)$) on $H^1_A(X; adP)$ has the form $\underline{\mathbb{R}}^k \oplus \mathbb{C}^l$, where the action on $\mathbb{C}^l$ has weight two. Again, this has no effect on the invariant functions so we ignore this fact, too.

Now consider rank two connections ($G = U(2)$ or $SU(2)$) on a $\mathbb{Z}$HS X. Since the only flat $U(1)$ connection on X, up to gauge, is the trivial connection, a flat G connection cannot be of type $1+1$, and there is only one 1^2 flat $SU(2)$ connection (the rank two trivial connection θ). Finally, since $H^1_\theta(X; adP) \cong H^1(X;\mathbb{R}) \otimes Lie(G) = 0$, this connection is isolated.

Next, consider rank two connections on a $\mathbb{Q}$HS X. There are finitely many nontrivial $U(1)$ flat connections up to gauge, so now there do exist nontrivial 1^2 and $1+1$ rank two connections. Again, for a 1^2 connection A, $H^1_A(X; adP) = 0$. For the $1+1$ connections, $H^1_A(X; adP)$ is isomorphic (under the $stab(A) = U(1)$ action) to $\mathbb{C}^l$.

We now consider rank three connections on a $\mathbb{Z}$HS X. Although in general the $U(3)$ connection types are $\{3, 2+1, 1+1+1, 1+1^2, 1^3\}$, the homology restriction only allows types 3, $2+1$ and the unique rank three trivial connection (and again the gauge orbit of the trivial connection is isolated). The $2+1$ connections have $U(1)$ stabilizer in $SU(3)$; the stabilizer is larger in $U(3)$, but this distinction is again unimportant.

On a $\mathbb{Q}$HS, there can be nontrivial 1^3 flat connections, but all of these are isolated. There can also be 1^2+1 and $1+1+1$ flat connections. There is one simplifying observation to be made about 1^2+1 connections, however. For a 1^2+1 connection flat connection A, $stab(A)$ is (effectively) $U(2)$. Since $U(2) \times 1 \subset U(3)$ acts on $su(3)$ in such a way that it is isomorphic to $su(2) \oplus \underline{\mathbb{R}} \oplus \mathbb{C}^2$, $H^1_A(X; adP)$ is isomorphic as a $U(2)$ vector space to $\underline{\mathbb{R}}^k \oplus (\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l) \oplus (su(2) \otimes \underline{\mathbb{R}}^m)$. However, the $su(2)$ component must vanish by the same argument as for the 1^2 connections above. Thus the nontrivial part of the $U(2)$ action on $H^1_A(X; adP)$ is isomorphic to $\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l$.

Finally, consider rank four connections. In general, rank four connections have the following types:

$$\{4, 3+1, 2+2, 2^2, 2+1^2, 2+1+1, 1+1+1+1, 1^2+1+1, 1^2+1^2, 1^3+1, 1^4\}.$$

Again, the reducible (nontrivial) flat connections on a $\mathbb{Z}$HS are limited to types $3+1$ and $2+2$, which are analogous to the reducible $SU(3)$ connections mentioned above and discussed in detail in [3], and types 2^2 and $2+1^2$. For a 2^2 flat connection, the $stab(A) = U(2)$ action on $H^1_A(X; adP)$ is equivalent to $\underline{\mathbb{R}}^k \oplus (su(2) \otimes \underline{\mathbb{R}}^l)$. It is equivalent but easier to consider $H = SO(3)$ acting on $\underline{\mathbb{R}}^k \oplus (\mathbb{R}^3 \otimes \underline{\mathbb{R}}^l)$.

Finally, for $2+1^2$ flat connections on a 3-manifold, $H^1_A(X; adP)$ is equivalent to $\underline{\mathbb{R}}^k \oplus (\mathbb{C}^2 \otimes \underline{\mathbb{R}}^l) \oplus (su(2) \otimes \underline{\mathbb{R}}^m)$. Again the $su(2)$ component is identified with $H^1(X; \mathbb{R}) \otimes su(2)$ and hence is trivial.

Hence if X is a $\mathbb{Q}$HS and $G = SU(3)$ or $U(3)$, or if X is a $\mathbb{Z}$HS and $G = SU(4)$ or $U(4)$, the the $stab(A)$ actions on $H^1_A(X; adP)$ all effectively have the types listed in the table before Theorem 1. Note that each of these representations was addressed in the abundance extension results of Section 5.

We now state the state the theorem which is the main goal of this paper. Let X be a closed, oriented 3-manifold. Choose a collection of thickened loops $\gamma_1, \ldots, \gamma_N$ in X generating $\pi_1 X$. Fix $s \geq 6$ and use $C^s_G(G^N)$ to define a space of admissible functions $\mathcal{F}$, with the C^s Banach norm.

Theorem 65 *Assume that either*

(A) X is a $\mathbb{Q}$HS and $G = SU(3)$ or $U(3)$, or
(B) X is a $\mathbb{Z}$HS and $G = SU(4)$ or $U(4)$.

There exists a $\delta > 0$ and a residual subset of $B_\delta(\mathcal{F})$ for which the perturbed flat moduli space $\mathcal{M}_f(X)$ is a finite collection of gauge orbits ($\mathrm{cs} + f$ is Bott-Morse on $\mathcal{A}/\mathcal{G}$ with finitely many isolated critical orbits). For two such perturbations f_0 and f_1, generic paths in $B_\delta(\mathcal{F})$ from f_0 to f_1 have the property that the parameterized flat moduli space is a compact union of arcs. The individual *arcs are compact except for open ends that limit to distinct points in a more reducible stratum of the parameterized critical set, as follows.*

In Case (A), type 3 arcs can limit only to type $2+1$ *points; type* $2+1$ *arcs can limit to* $1+1+1$ *or* $1+1^2$ *points, and the* $1+1+1$ *and* $1+1^2$ *strata of the parameterized moduli space are compact. (See Figure 1.) In Case (B), type* 4 *arcs can limit to* $2+2$ *or* $3+1$ *points;* $2+2$ *components can limit to* 2^2 *points; and* $3+1$ *components can limit to* $2+1^2$ *points. (See Figure 2.)*

For $\delta > 0$ *sufficiently small, all the observations about the flat moduli space preceding this theorem continue to hold for the perturbed moduli space.*

PROOF. For each flat connection A, there is a $\delta_A > 0$ and neighborhoods $B_1 \subset H^1_A(X; adP)$ and $B_2 \subset H^1_A(X; adP)^\perp \cap L^2_1\mathcal{X}_A$ such that the parameterized Kuranishi model describes makes sense in $B_1 \times B_2 \times B_{\delta_A}(\mathcal{F})$. Since Proposition 62 guarantees that $\mathcal{T}^A : \mathcal{F} \to C^5_{stab(A)}(B_1)$ is fourth order abundant at $(0,0) \in B_1 \times B_{\delta_A}(\mathcal{F})$, we can apply the extension results in Section 5 to find smaller neighborhoods where second order abundance holds (except on the principal orbit stratum where we at least have first order abundance). Note that these neighborhoods are contained in the slices through flat connections, so no connection in such a neighborhood is more reducible that the types of the (nontrivial) flat connections listed above. In the case of a $2+1^2$ flat connection, we can also choose the neighborhood around it small enough, and δ_A small enough, that no perturbed flat $2+1^2$ connection in the neighborhood has a nontrivial $su(2)$ component in the cohomology.

Since the flat moduli space is compact, we can cover it with (the image under the quotient map $p : \mathcal{A} \to \mathcal{B}$ of) a finite collection of such open neighborhoods. Set δ equal to the minimum of the corresponding δ_A's. Taubes shows in [20] that by choosing $\delta > 0$ sufficiently small, we can insure that the perturbed flat moduli space $\mathcal{M}_f$ for $\|f\| < \delta$ lies in union of this open cover. Taubes also shows that the perturbed flat moduli space is compact for any admissible function f.

By applying the local results of Section 4 to the individual neighborhoods, and then intersecting the residual subsets of perturbations and paths we arrive at global versions of these theorems for the perturbed flat moduli space.

Finally, let us address the question of which types of perturbed flat connections can appear out of a given type. The answer to this question is summarized in Figures 1 and 2. To see that no other possibilities exist, suppose a generic path of perturbations f_t is chosen so that the kernel of the Hessian of $\mathcal{T}^{A_i}(f_t)$ at a critical point $a_0 \in H^1_{A_i}(X; adP)$ is never larger than one irreducible representation of the stabilizer of a_0. (This is part of the jet condition described in Theorem 46.)

The orbit type (under the $stab(a_0) \subset stab(A_i)$ action) of a nonzero vector in this kernel is determined completely by the orbit type of a_0 under $stab(A_i)$ (which is determined by that of $A_i{+}a_0$ under the gauge group). For example, if $A_i{+}a_0$ is a $2{+}1^2$ connection then the $stab(A_i{+}a_0)$ action on the kernel is either trivial (and the nearby solutions are of the same orbit type) or is effectively $U(2)$ acting on $\mathbb{C}^2$. Note that the $\mathbb{C}^2$ here corresponds to "intertwining" the $U(2)$ connection with a 1-dimensional subspace of the "1^2" part. Any nonzero vector in $\mathbb{C}^2$ has isotropy in $U(2)$ isomorphic to $U(1)$, and the corresponding connection is a $3 + 1$ type connection. $\square$

In a future paper, we will use this result to define an $SU(4)$ invariant of $\mathbb{Z}$HS's, and also a $U(3)$ or $SU(3)$ invariant of $\mathbb{Q}$HS's, which are analogs of the $SU(3)$ Casson invariant defined in [4]. In the rank four case, irreducible (i.e. type 4) points should be counted with sign determined by the spectral flow. The $2+2$ and $3{+}1$ strata should contribute correction terms analogous to that for $\tau_{SU(3)}$ defined in [4], and the other two strata must contribute "correction terms to the correction terms." Similarly, in the rank 3 case, secondary correction terms for the reducible strata of type $1^2 + 1$ and $1 + 1 + 1$ must be added to the τ definition in [4].

The importance of the complete catalog of bifurcations provided here is that one can now use a Cerf-theoretic argument to show the signed count of irreducible points plus the correction terms is invariant. To prove that this quantity is independent of the choice of perturbation, it is sufficient to demonstrate that none of the bifurcations in Figure 1 or Figure 2, and no compact cobordisms within any stratum, change this sum, following the approach taken in [3], [4] and [6].

To generalize the transversality results in Theorem 65 beyond the Cases (A) and (B) considered in this paper, one must generalize Theorem 53 to all the different groups $H = stab(A)$ and H representations $U = H^1_A(X; adP)$ for the all the different reduction types. In addition, the sphere transitivity property no longer holds for each of the H vector spaces. For example, a rank six connection of type 2^3 has stabilizer in $U(6)$ isomorphic to $U(3)$. One component of the first cohomology will correspond to the $U(3)$ adjoint representation on $su(3)$, which is irreducible but does not have the sphere transitivity property. In these circumstances, the bifurcations can be analyzed but they are more complicated than those described here, and higher degree abundance is necessary.

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