

## Final exam, Math 675

[Due at noon on December 18th]

**Instructions** In all the problems below  $\gamma$  is the circle which defines the disk in the Poincaré disk model for hyperbolic geometry. We refer to it simply as the “Poincaré circle”. Lines and lengths in this model are likewise referred to as “Poincaré lines” or “Poincaré lengths”.

You are allowed to use any and all theorems, propositions etc. from the textbook (and if you do, you should provide a reference to a page number) but you should not rely on facts found on the web (unless you supply a proof for them).

1. Let  $\gamma$  be a Poincaré circle in the Euclidean plane. An *ideal point* in this model is any point on  $\gamma$  (such points are not part of the hyperbolic model itself). Show that given any two distinct ideal points  $P$  and  $Q$ , there exist a unique Poincaré line whose end points are  $P$  and  $Q$ .
2. Let  $\gamma$  be the Poincaré circle in the Euclidean plane given by the equation  $x^2 + y^2 = 8$ . Let  $P = (2, 2)$  and  $Q = (-2, 2)$  be two ideal points (see exercise 1). Let  $\ell$  be the unique Poincaré line with end points  $P$  and  $Q$  and let  $X = (0, 0)$  (see figure 1). Find the angle of parallelism for  $X$  with respect to  $\ell$ .

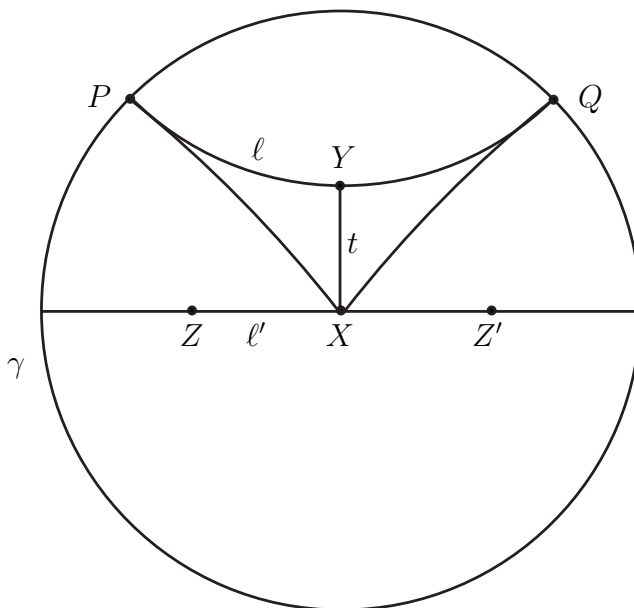


FIGURE 1. Given the ideal points  $P$  and  $Q$  on  $\gamma$ , let  $\ell$  be the Poincaré line determined by  $P$  and  $Q$ . Let  $t$  be the unique Poincaré line through  $X$  and perpendicular to  $\ell$  and let  $\ell'$  be the unique Poincaré line through  $X$  perpendicular to  $t$ . Pick points  $Z, Z'$  on  $\ell'$  with  $Z * X * Z'$ . Then the angle of parallelism of  $X$  with respect to  $\ell$  is either of the angles  $\angle ZXP$  or  $\angle Z'XQ$ .

3. Let  $\gamma$  be the Poincaré circle given by the equation  $x^2 + y^2 = 1$  and let  $\delta$  be the circle  $(x - \sqrt{2})^2 + y^2 = 1$ . Let furthermore  $A$  and  $B$  be the points in the interior of  $\gamma$  given by  $A = (\sqrt{2} - 1, 0)$  and  $B = (1/2, 1/2)$ . Let  $A'$  and  $B'$  be the inverses of  $A$  and  $B$  with respect to inversion in  $\delta$ .
- Show that  $\delta$  is perpendicular to  $\gamma$ .
  - Show by an explicit coordinate calculation that the Poincaré lengths  $d(AB)$  and  $d(A'B')$  are equal.
4. Let  $\gamma$  be a Poincaré disk and let  $\mathfrak{D}$  be its interior. Let  $\mathcal{F} : \mathfrak{D} \rightarrow \mathfrak{D}$  be a transformation of  $\mathfrak{D}$  which preserves Poincaré length, that is

$$d(\mathcal{F}(A), \mathcal{F}(B)) = d(A, B)$$

for any two points  $A, B \in \mathfrak{D}$ . A *fixed point* of  $\mathcal{F}$  is a point  $A \in \mathfrak{D}$  with  $\mathcal{F}(A) = A$ . Show that if  $\mathcal{F}$  has two fixed points then it actually has infinitely many fixed points. *Hint:* Show that if  $A$  and  $B$  are fixed points of  $\mathcal{F}$  then every point on the Poincaré line through  $A$  and  $B$  is a fixed point of  $\mathcal{F}$ .

5. Let  $\triangle ABC$  be a triangle in hyperbolic space. Show that the three angle bisectors of  $\triangle ABC$  are concurrent.
6. Prove or disprove the following claim: Given any three positive numbers  $a, b$  and  $c$  with

$$a < b + c$$

$$b < a + c \quad (\text{think “triangle inequality”})$$

$$c < a + b$$

there exists a triangle  $\triangle ABC$  in hyperbolic geometry with

$$d(AB) = c$$

$$d(AC) = b$$

$$d(BC) = a$$

7. Recall the isomorphism  $F$  from the Klein model for hyperbolic geometry to the Poincaré disk model (see also page 236 of the textbook): Let  $\kappa$  be the circle from the Klein model thought of as lying in the Euclidean plane which in turn we think of as lying in 3-dimensional Euclidean space. Let  $S$  be a sphere of the same radius as  $\kappa$  which at its south pole touches the Euclidean plane at the center of  $\kappa$ . Let  $\gamma$  be the circle gotten from  $\kappa$  by projecting  $\kappa$  upwards onto  $S$  (thereby covering the equator of  $S$ ) and then stereographically projecting back onto the Euclidean plane from the north pole  $N$  of  $S$ . A point  $A$  from the interior of  $\kappa$  is mapped to a point  $F(A) = A''$  in the interior of  $\gamma$  by first mapping it upwards onto the southern hemisphere of  $S$  (resulting in the point  $A'$ ) and then stereographically projecting it from the north pole back into the Euclidean plane, see figure 2 below.
- If the radius of  $\kappa$  is  $r$ , what is the radius of  $\gamma$ ?
  - Introduce the complex coordinate  $z = x + iy$  into the Euclidean plane so that the center of the coordinate system coincides with the center of  $\kappa$ .

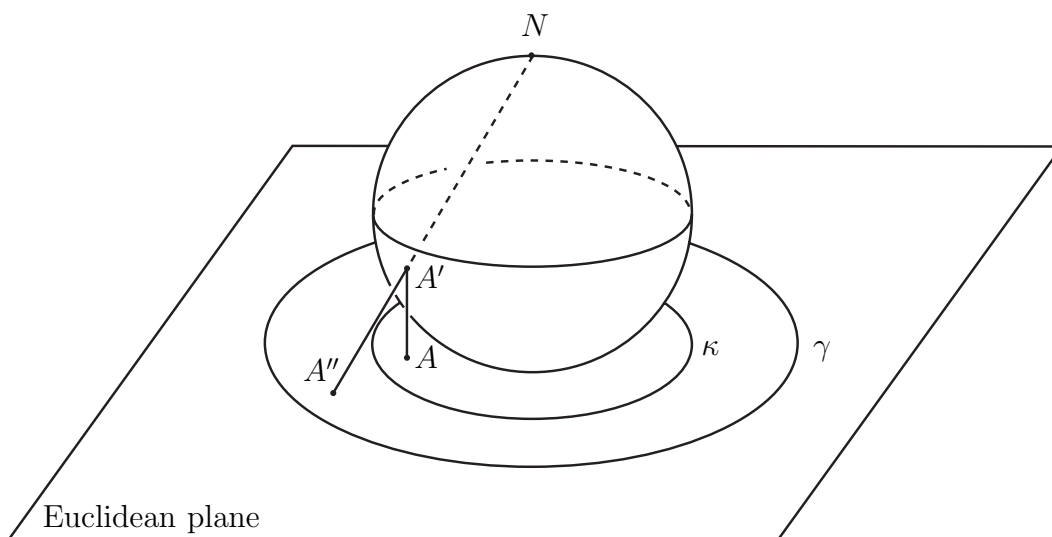


FIGURE 2. A visual representation of the isomorphism from the Klein model  $\kappa$  to the Poincaré disk model  $\gamma$  which sends a point  $A$  from the interior of  $\kappa$  to the point  $A''$  in the interior of  $\gamma$ .

Show that

$$F(z) = \frac{2rz}{r + \sqrt{r^2 - |z|^2}}$$

where  $|z|^2 = x^2 + y^2$ .

- c) Pick two Beltrami-Klein lines  $\ell_1$  and  $\ell_2$  (i.e. two open chords) of which one, say  $\ell_1$ , is a diameter. Let  $m_1$  and  $m_2$  be their images under  $F$  in the Poincaré disk. Show that  $\ell_1 \perp \ell_2$  if and only if  $m_1 \perp m_2$ .

*Hint:* Pick the coordinate system you use so that  $\ell_2$  (or  $m_2$ ) take a particularly simple form. For example you might want to arrange that  $\ell_2$  is parallel to either the  $x$  or  $y$ -axis or your coordinate system.