Midterm exam, Math 675

[Due on November 1st]

1. A quadrilateral $\square ABCD$ is called a *Saccheri quadrilateral* if $AB \cong CD$ and if the angles $\angle B$ and $\angle C$ are right angles, see figure 1. Show that in a Saccheri quadrilateral the congruency relation $\angle A \cong \angle D$ holds.

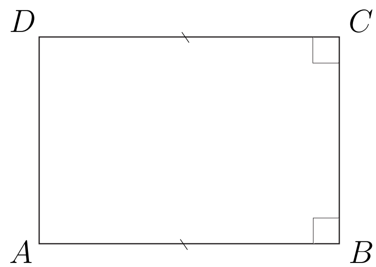

FIGURE 1. A Saccheri quadrilateral.

2. Show that any model of neutral geometry has to have infinitely many points.
3. Show that in neutral geometry there is an equivalence between the two statements:
   (a) Hilbert's parallel postulate holds.
   (b) The angle sum in every triangle is 180^o.
4. Prove that in neutral geometry, a line cannot be contained in the interior of an angle.
5. Prove that in neutral geometry every angle has a unique bisector. This is the content of proposition 4.4a.
6. Prove that in neutral geometry, the diagonals of a convex quadrilateral intersect.
7. Prove that nonisoceles triangles exist? Which axioms of neutral geometry do you need to use to prove this existence?
8. Let $\triangle ABC$ be a right triangle with the right angle at C and with the angle $\angle A$ being acute. Create a new right triangle $\triangle AB'C'$ whose angle $\angle A$ coincides with $\angle A$ of the first triangle but whose hypotenuse has been double in length i.e. $\overline{AB'} = 2 \cdot \overline{AB}$, see figure 2.

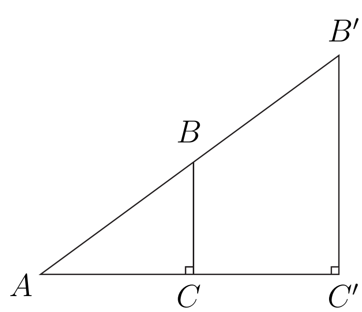

FIGURE 2

   (a) Show that $\overline{B'C'} \geq 2 \cdot \overline{BC}$.
   (b) Show that $\overline{AC'} \leq 2 \cdot \overline{AC}$.

9. A *parallelogram* is a quadrilateral whose opposite sides lie on parallel lines. Given a quadrilateral $\square ABCD$ show that in neutral geometry
   (a) Opposite sides of the parallelogram are congruent.
   (b) Show that a parallelogram is a rectangle if and only if its diagonals are congruent.
10. Prove the converse of *Euclid V* in neutral geometry. That is, in the context of neutral geometry show that if ℓ and ℓ' two lines which intersect at A and t is a common transversal for ℓ and ℓ', then the angle sum of the two interior angles of t which lie on the same side of t as A (angles $\angle 1$ and $\angle 2$ in figure 3), have measures which add up to less than 180^o.

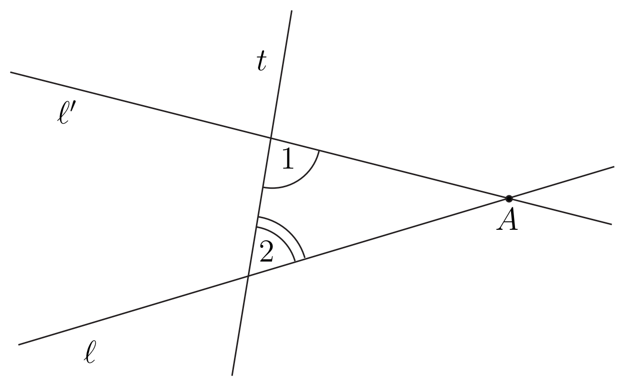

FIGURE 3

11. Show that in Euclidean geometry (i.e. neutral geometry + Hilbert's parallel postulate) for every triple of positive numbers α, β, λ with $\alpha + \beta < 180^o$ there is a triangle $\triangle ABC$ with

$$(\angle A)^\circ = \alpha \qquad (\angle B)^\circ \cong \beta \qquad \overline{AB} = \lambda$$

Is it possible to find such triangles in neutral geometry?