CHAPTER 7

Connectedness

7.1. Connected topological spaces

**Definition 7.1.** A topological space \((X, \mathcal{T}_X)\) is said to be **connected** if there is no continuous surjection \(f : X \to \{0, 1\}\) where the two point set \(\{0, 1\}\) is given the discrete topology. Otherwise \(X\) is said to be **disconnected**.

**Theorem 7.2.** Let \((X, \mathcal{T}_X)\) be a topological space. Then the following are equivalent:

(a) \(X\) is connected.

(b) The only open and closed subsets of \(X\) are the empty set and \(X\) itself.

(c) If \(X = A \cup B\) with \(A, B\) disjoint and open sets, then \(A = \emptyset\) or \(B = \emptyset\).

**Proof.** (a)⇒(b) If \(A \subset X\) is both open and closed, then so is \(X - A\). If both of these were nonempty, as would be the case if \(A \neq \emptyset, X\), then we could define the surjective function \(f : X \to \{0, 1\}\) as

\[
 f(x) = \begin{cases} 
 0 & \text{if } x \in A \\
 1 & \text{if } x \in X - A 
\end{cases}
\]

An easy check reveals that \(f\) is continuous, contradicting the assumption from part (a) that \(X\) is connected.

(b)⇒(c) If \(X = A \cup B\) with \(A\) and \(B\) open, then \(A\) and \(B\) are also closed (since \(B\) being open implies that \(A = X - B\) is closed and similarly for \(B\)). According to part (b), one of \(A\) or \(B\) is then the empty set.

(c)⇒(a) If \(f : X \to \{0, 1\}\) is a surjective map, then \(f^{-1}\{\{0\}\}\) and \(f^{-1}\{\{1\}\}\) are disjoint subsets of \(X\) that are open and closed (since \(f\) is continuous and since \(\{0\}\) and \(\{1\}\) are open and closed subsets of \(\{0, 1\}\)). By surjectivity of \(f\), they are also both non-empty contradicting part (c). Therefore, no continuous surjection \(f : X \to \{0, 1\}\) can exist. □

**Lemma 7.3.** Let \(f : X \to Y\) be a continuous function between topological space. If \(X\) is connected then so is \(f(X)\). In particular, the property of “being connected” is a topological invariant.

**Proof.** Suppose that \(f(X) = A \cup B\) with \(A, B \subset f(X)\) disjoint and open sets. Then \(f^{-1}(A)\) and \(f^{-1}(B)\) are open subsets of \(X\) with \(X = f^{-1}(A) \cup f^{-1}(B)\). By connectedness of \(X\), one of \(f^{-1}(A)\) or \(f^{-1}(B)\) has to be the empty set, say \(f^{-1}(B)\). But then \(B = \emptyset\) showing that \(f(X)\) is connected.

If \(X\) and \(Y\) are homeomorphic, each of them is the image of the other under a continuous map and so they are both connected or both disconnected. □
Example 7.4. If \( X = \{ x \} \) is a set with only a single point, then it is connected for there is no surjection from a set with one element onto a set with two elements.

Example 7.5. The excluded point topology makes \( \mathbb{R} \) into a connected space. This follows from the observation that if \( \mathbb{R} = A \cup B \) with \( A, B \) open, then if \( p \in A \) we get \( A = \mathbb{R} \) and if \( p \in B \) then \( B = \mathbb{R} \). In either case, \( A \) and \( B \) can only be disjoint if one of them is the empty set.

Example 7.6. The set of rational number \( \mathbb{Q} \subset \mathbb{R} \) with its relative Euclidean topology, is disconnected. This can be seen by setting \( A = \langle -\infty, \sqrt{2} \rangle \cap \mathbb{Q} \) and \( B = \langle \sqrt{2}, \infty \rangle \cap \mathbb{Q} \). Then \( A, B \subset \mathbb{Q} \) are two open and disjoint sets and \( \mathbb{Q} = A \cup B \).

Example 7.7. If \( X \) is any set with at least two elements, then \((X, T_{dis})\) is disconnected. On the other hand, for any set \( Y \), the space \((Y, T_{indis})\) is connected (here \( T_{dis} \) is the discrete and \( T_{indis} \) is the indiscrete or trivial topology).

Example 7.8. With the finite complement topology, \( \mathbb{R} \) is a connected space since there are no non-empty disjoint open subsets of \( \mathbb{R} \). The same holds true with the countable complement topology.

In the statement of the next theorem we shall use the word *interval* to mean any of \( [a, b) \), \( [a, b] \), \( (a, b) \) or \( [a, b] \). We allow \( a \) and \( b \) at the open ends of these intervals to be \( \pm \infty \), e.g. the sets \( (-\infty, 2] \) and \( (0, \infty) \) will also be called intervals. In particular, intervals are precisely the convex subsets of \( \mathbb{R} \). With this understanding, the next theorem characterizes the connected subsets of the Euclidean line.

**Theorem 7.9.** Consider the Euclidean line \((\mathbb{R}, T_{Eu})\). A nonempty subspace \( A \subset \mathbb{R} \) is connected if and only if it is an interval.

**Proof.** Let \( A \) be a nonempty subspace of \( \mathbb{R} \). If \( A \) contains only one point then it is connected according to example 7.4. Suppose that \( A \) contains at least two points, say \( x \) and \( y \), and suppose that \( x < y \). If \( z \in \mathbb{R} \) is any point with \( x < z < y \) then \( z \) has to belong to \( A \). For if not, then \( A \) could be written as the union
\[
A = (\langle -\infty, z \rangle \cap A) \cup (\langle z, \infty \rangle \cap A)
\]
with both sets on the right hand side being open and nonempty (and clearly disjoint) contradicting the connectedness assumption of \( A \). Therefore \( z \in A \). Let \( \ell = \inf A \) and \( L = \sup A \), then \( A \) equals an interval from \( \ell \) to \( L \), the boundaries may be included or not and may be infinite or not.

**Theorem 7.10.** Let \((X, T_X)\) be a topological space and let \( Y \) be a subspace of \( X \). If \( Y \) is connected and \( Z \subset X \) is any set with \( Y \subset Z \subset \bar{Y} \), then \( Z \) is also connected.

**Proof.** Suppose first that \( Z = \bar{Y} \). To show that \( \bar{Y} \) is connected let us write \( \bar{Y} = A \cup B \) with \( A, B \) open and disjoint subsets of \( \bar{Y} \). Then \( A_Y = A \cap Y \) and \( B_Y = B \cap Y \) are open and disjoint subsets of \( Y \) with \( Y = A_Y \cup B_Y \). By connectedness of \( Y \), one of \( A_Y \) or \( B_Y \) has to be the empty set, say \( B_Y = \emptyset \). Then \( A_Y = Y = A \cap \bar{Y} \) showing that \( Y \subset \bar{Y} - B \). Since \( B \) is open, the set \( \bar{Y} - B \) is closed and contains \( Y \) and
so \( \bar{Y} \subset Y - B \) (since \( \bar{Y} \) is the smallest closed set containing \( Y \)) showing that \( B = \emptyset \). Consequently \( \bar{Y} \) is connected.

Now let \( Z \subset \bar{Y} \) be any set with \( Y \subset Z \) and write again \( Z = A \cup B \) with \( A, B \) open and disjoint subsets of \( Z \). Then, as in the previous paragraph, \( A_Y = A \cap Y \) and \( B_Y = B \cap Y \) are open and disjoint subsets of \( Y \) with \( Y = A_Y \cup B_Y \). Again, by connectedness of \( Y \), one of \( A_Y \) or \( B_Y \) is empty, say \( B_Y = \emptyset \). Let \( B' \subset \bar{Y} \) be an open set with \( B = Z \cap B' \), then \( Y \subset \bar{Y} - B' \). Since \( B' \) is open, \( \bar{Y} - B' \) is closed and contains \( Y \) and so \( \bar{Y} \subset Y - B' \) showing that \( B' = \emptyset \). But then \( B = Z \cap B' = Z \cap \emptyset = \emptyset \) and therefore \( Z \) is connected. \( \square \)

**Corollary 7.11.** The closure of a connected subspace is itself connected. If a space \( X \) has a connected dense subset, then \( X \) is also connected.

**Example 7.12.** Consider the included point topology \( T_p \) on \( \mathbb{R} \). The closure of \( \{p\} \) is all of \( \mathbb{R} \) (since \( \mathbb{R} \) is the only closed set containing \( p \)) and since a set with only one element is always connected (example 7.4), it follows from theorem 7.10 that \( (\mathbb{R}, T_p) \) is connected.

**Definition 7.13.** Let \( (X, T_X) \) be a topological space.

(a) A connected component \( U \) of \( X \) is any maximal connected subset of \( X \). Said differently, \( U \) is a connected component of \( X \) if, whenever \( V \) is a connected subspace of \( X \) and \( U \subset V \), then \( U = V \).

(b) We say that \( X \) is totally disconnected if every connected component of \( X \) consists of just a single point.

**Lemma 7.14.** Let \( (X, T_X) \) be a topological space and \( Y_i \subset X, i \in I \) be a family of connected subspaces. If \( \cap_{i \in I} Y_i \neq \emptyset \) then \( \cup_{i \in I} Y_i \) is connected.

**Proof.** Let \( p \in \cap_{i \in I} Y_i \) be any point and let \( A, B \subset \cup_{i \in I} Y_i \) be two open and disjoint subsets of \( \cup_{i \in I} Y_i \) with \( \cup_{i \in I} Y_i = A \cup B \). Then \( A_i = A \cap Y_i \) and \( B_i = B \cap Y_i \) are open and disjoint subsets of \( Y_i \) with \( Y_i = A_i \cup B_i \). By connectedness assumption of \( Y_i \), one of \( A_i \) or \( B_i \) needs to be the empty set. Suppose that for a given index \( i_0 \in I \) we had \( B_{i_0} = \emptyset \) and notice that then \( p \in A_{i_0} \subset A \). For any other index \( i \in I \), we cannot have \( A_i = \emptyset \) since that would imply that \( p \in B_i \subset B \), a contradiction since we already found that \( p \in A \). Therefore we obtain that \( B_i = \emptyset \) for every \( i \in I \). But then \( B = \cup_{i \in I} B_i = \emptyset \) showing that \( \cup_{i \in I} Y_i \) is connected. \( \square \)

**Theorem 7.15.** Let \( (X, T_X) \) be a topological space.

(a) \( X \) is the disjoint union of its connected components.

(b) Every connected component of \( X \) is a closed subset of \( X \). In particular, if \( X \) has finitely many components, then each component is both open and closed.

(c) \( X \) is connected if an only if it has precisely one connected component.

**Proof.** (a) Let \( x \in X \) be any point and let \( U_x \) be the subset of \( X \) obtained by \( U_x = \cup_{U \in \mathcal{U}} U \) with \( \mathcal{U} = \{W \subset X \mid x \in W \text{ and } W \text{ is connected}\} \)
By lemma 7.14, $U_x$ is connected and it is clearly a maximal connected subset of $X$. Thus every point $x \in X$ belongs to a connected component of $X$ showing that $X$ is the union of its connected components.

Let $U, V \subset X$ be two different connected components of $X$. If we had $U \cap V \neq \emptyset$ then $U \cup V$ would be a connected space (again according to lemma 7.14), a contradiction to maximal property of both $U$ and $V$. Thus $U \cap V = \emptyset$.

(b) If $U$ is a component of $X$ then $\overline{U}$ is also connected (according to theorem 7.10) and so by maximality, $U = \overline{U}$. Accordingly, $U$ is closed.

c) This follows directly from the definition. If the only component of $X$ is $U$ then, since every component is connected, so is $X$. If $X$ is connected, let $U_i, i \in I$ be its components. Then $X$ is a connected set containing $U_i$ and so by the maximality property of $U_i$, we must have $X = U_i$ for all $i \in I$. □

We note that part (c) of the above theorem cannot in general be improved by claiming that connected components of $X$ are always open subsets (however, see corollary 7.40 for more information). Here is an example demonstrating this.

Example 7.16. Consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ equipped with the relative Euclidean topology. If $Y \subset \mathbb{Q}$ is any subset with at least two elements, say $y_1, y_2 \in Y$ with $y_1 < y_2$, then

$$A = \langle -\infty, \lambda \rangle \cap Y \quad \text{and} \quad B = \langle \lambda, \infty \rangle \cap Y$$

with $\lambda \in \langle y_1, y_2 \rangle$ arbitrary, are two disjoint, open and non-empty subsets of $Y$ with $Y = A \cup B$. Thus, any such $Y$ is disconnected showing that the only connected subsets of $\mathbb{Q}$ are sets with only one element. Thus $\mathbb{Q}$ is totally disconnected and its components are $\{q\}, q \in \mathbb{Q}$. While $\{q\}$ is a closed subset of $\mathbb{Q}$ (as claimed by part (c) of theorem 7.15), its is not open. This shows that part (c) from theorem 7.15 cannot be strengthened in general.

Example 7.17. Consider the lower limit topology $\mathcal{T}_{ll}$ on $\mathbb{R}$. The connected components ($\mathbb{R}, \mathcal{T}_{ll}$) are the single point sets $\{x\}, x \in \mathbb{R}$ since whenever $A \subset \mathbb{R}$ has at least two elements, say $a, b \in A$ with $a < b$, then $A$ can be written as the following union of two disjoint and non-empty open subsets of $A$:

$$A = (\langle -\infty, b \rangle \cap A) \cup ([b, \infty) \cap A)$$

This shows that any such $A$ is disconnected. Consequently, ($\mathbb{R}, \mathcal{T}_{ll}$) is completely disconnected.

Example 7.18. Consider the Euclidean plane ($\mathbb{R}^2, \mathcal{T}_{Eu}$) and let $Y$ be its subspace given by

$$Y = \{(x, y) \in \mathbb{R}^2 \mid xy = 0 \text{ and } (x, y) \neq (0, 0)\}$$
7.2. PATH CONNECTEDNESS

Geometrically, $Y$ is the union of the $x$-axis with the $y$-axis with the origin removed. Then $Y$ has 4 connected components and they are:

- $U_+ = \{(x,0) \in \mathbb{R}^2 \mid x > 0\}$ = Positive half of the $x$-axis.
- $U_- = \{(x,0) \in \mathbb{R}^2 \mid x < 0\}$ = Negative half of the $x$-axis.
- $V_+ = \{(0,y) \in \mathbb{R}^2 \mid y > 0\}$ = Positive half of the $y$-axis.
- $V_- = \{(0,y) \in \mathbb{R}^2 \mid y < 0\}$ = Negative half of the $y$-axis.

To see that these are indeed the components of $Y$, note firstly that each of $U_\pm$ and $V_\pm$ is homeomorphic to $\mathbb{R}$ and therefore connected according to theorem 7.9. To see that $U_+$ is a maximal connected subspace of $Y$, let $A, B \subset \mathbb{R}^2$ be the open and disjoint subsets given by

$$A = \{(x,y) \in \mathbb{R}^2 \mid x < |y|\} \quad \text{and} \quad B = \{(x,y) \in \mathbb{R}^2 \mid x > |y|\}$$

Then $U_+ \subset A$ and $U_- \cup V_+ \cup V_- \subset B$. If $U_+$ were not a maximal connected subspace of $Y$ and were contained in a larger connected subset $\bar{U} \subset Y$, then $A \cap \bar{U}$ and $B \cap \bar{U}$ would be disjoint, open and non-empty subsets of $\bar{U}$ with $\bar{U} = (A \cap \bar{U}) \cup (B \cap \bar{U})$, a contradiction to the connectedness of $\bar{U}$. Consequently, $U_+$ is a component of $Y$, the cases of $U_-, V_+, V_-$ are handled analogously.

7.2. Path connectedness

**Definition 7.19.** Let $(X, \mathcal{T}_X)$ be a topological space.

(a) A path in $X$ is a continuous function $\alpha : [0, 1] \to X$ where $[0, 1]$ is equipped with the relative Euclidean topology. The points $\alpha(0), \alpha(1)$ are called the initial (or starting) and terminal (or ending) points of $\alpha$. We will say that $\alpha$ joins $x$ to $y$ or that $x$ is joined to $y$ by $\alpha$.

(b) If $\alpha : [0, 1] \to X$ is a path in $X$ then we shall call the path $\bar{\alpha} : [0, 1] \to X$ given by $\bar{\alpha}(t) = \alpha(1-t)$, the inverse path of $\alpha$.

(c) Given two paths $\alpha, \beta : [0, 1] \to X$ with $\alpha(1) = \beta(0)$, we define the product path $\alpha \cdot \beta : [0, 1] \to X$ of $\alpha$ and $\beta$ as

$$\alpha \cdot \beta(t) = \begin{cases} 
\alpha(2t) & ; \ t \in [0, 1/2] \\
\beta(2t-1) & ; \ t \in [1/2, 1]
\end{cases}$$

A couple of remarks are in order. If $x$ is joined to $y$ by means of a path $\alpha$, then $y$ is joined to $x$ by the path $\bar{\alpha}$. The function $\alpha \cdot \beta$ from part (c) of definition 7.19 is continuous by lemma 3.13. If $\alpha$ joins $x$ to $y$ and $\beta$ joins $y$ to $z$ then $\alpha \cdot \beta$ joins $x$ to $z$.

With this in mind, we are ready to introduce the main concept of this section.

**Definition 7.20.** A topological space $X$ is called path connected if for every two points $x, y \in X$ there exists a path $\alpha$ in $X$ with $\alpha(0) = x$ and $\alpha(1) = y$.

**Lemma 7.21.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological space and let $f : X \to Y$ be a continuous function. If $X$ is path connected then so is $Y$. In particular, path connectedness is a topological invariant.
Proof. Given two points \( x, y \in f(X) \), let \( a, b \in X \) be points with \( f(a) = x \) and \( f(b) = y \). Let \( \alpha : [0, 1] \to X \) be a path connecting \( a \) to \( b \), then \( f \circ \alpha \) is a path in \( Y \) connecting \( x \) to \( y \).

If \( X \) and \( Y \) are homeomorphic spaces then they are each the image of the other under the homeomorphism between them. Consequently they are either both path connected or neither of them is. □

Theorem 7.22. If \( X \) is path connected then it is connected. The converse is not always true (see example 7.25 below).

Proof. Suppose \( X \) were not connected. Then we could write \( X = A \cap B \) with \( A, B \) two open, disjoint and non-empty sets. Let \( a \in A \) and \( b \in B \) be any two points and let \( \alpha : [0, 1] \to X \) be a path joining \( a \) to \( b \). Then the sets

\[
A' = \alpha^{-1}(A) \quad \text{and} \quad B' = \alpha^{-1}(B)
\]

are both open (since \( \alpha \) is continuous), non-empty (since \( 0 \in A' \) and \( 1 \in B' \)) and disjoint (since \( A \) and \( B \) are disjoint) subsets of \([0, 1]\). This implies that \([0, 1]\) is disconnected, a contradiction to theorem 7.10. Therefore \( X \) must be connected. □

Example 7.23. Euclidean \( n \)-dimensional space \((\mathbb{R}^n, \mathcal{T}_{Eu})\) is path connected and hence also connected. Given any two points \( x, y \in \mathbb{R}^n \), the path \( \alpha : [0, 1] \to \mathbb{R}^n \) given by \( \alpha(t) = x + t(y - x) \) starts at \( x \) and ends at \( y \).

Example 7.24. For any point \( x \in \mathbb{R}^n \) and for any \( r > 0 \), the ball \( B_x(r) \) is path connected (and thus also connected). This is seen by observing that every point \( y \in B_x(r) \) can be connected to \( x \) via the path \( \alpha_y : [0, 1] \to B_x(r) \) defined by \( \alpha_y(t) = y + t(x - y) \). Given any pair of points \( y_1, y_2 \in B_x(r) \), the path \( \alpha_{y_1} \cdot \alpha_{y_2} \) connects \( y_1 \) to \( y_2 \).

Example 7.25. (Of a connected but not path connected space) Let \( X \) be the topologists sine curve from example 2.18, i.e. let \( X \) be the subspace of the Euclidean plane \((\mathbb{R}^2, \mathcal{T}_{Eu})\) given by

\[
X = \{(x, \sin \frac{1}{x}) \mid x \in (0, 1]\} \cup \{(0) \times [0, 1]) \} \subset \mathbb{R}^2
\]

Note that the set \( Y_1 = \{(x, \sin(1/x)) \mid x \in (0, 1]\} \subset X \) is homeomorphic to \((0, 1]\) and is thus connected according to theorem 7.9. According to theorem 7.10 the closure \( \bar{Y}_1 \) of \( Y_1 \) is then also closed as is any set \( Z \subset \mathbb{R}^2 \) with \( Y_1 \subset Z \subset Y_1 \). We claim that \( X \) satisfies this double inclusion showing that it is connected. To show that \( X \subset \bar{Y}_1 \), it suffices to show that every element in \( Y_2 = \{(0) \times [0, 1]\} \) is the limit of a convergent sequence \( x_k \in Y_1 \). This is easily seen for if \((0, y) \in Y_2 \), let \( x_k = (t_k, y) \in Y_1 \) be any sequence with \( t_k \) a convergent sequence with limit 0. For example, choosing \( t_k \) as

\[
t_k = \frac{1}{t_1 + 2\pi k} \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad t_1 = \arcsin \left| \frac{x}{y} \right| (y)
\]

will do.
We will now show that $X$ is not path connected. If we suppose to the contrary that $X$ is path connected, then we can find a path $\alpha : [0, 1] \rightarrow X$ connecting $(0, 0)$ to $(1, \sin 1)$. Since the $y$-axis is a closed subset of $\mathbb{R}^2$, the set 

$$A = \alpha^{-1}(y\text{-axis})$$

is a closed subset of $[0, 1]$ and it is nonempty since it contains 0. Let $t_0 \in A$ be any point and suppose that $\alpha(t_0) = (0, b_0)$ for some $b_0 \in [0, 1]$. Consider the neighborhood $V$ of $(0, b_0)$ in $X$ given by

$$V = \left((-\frac{1}{2}, \frac{1}{2}) \times (b_0 - \frac{1}{2}, b_0 + \frac{1}{2})\right) \cap X$$

The choice of $\frac{1}{2}$ in the definition of $V$ is somewhat random, any number less than 2 will do. By continuity of $\alpha$, there exists a $\delta > 0$ so that $\alpha((t_0 - \delta, t_0 + \delta) \cap [0, 1]) \subset V$. Since $(t_0 - \delta, t_0 + \delta) \cap [0, 1]$ is path connected, so is its image (theorem ??), and it must therefore lie in the path connected component of $V$ that contains $\alpha(t_0)$. This however is simply $V \cap y$-axis (see figure 1) and so $(t_0 - \delta, t_0 + \delta) \cap [0, 1]$ is a subset of $A$. This shows that $A$ is both open and closed and so is therefore $B = [0, 1] - A$. Likewise, both $A$ and $B$ are nonempty seeing as they contain 0 and 1 respectively. This contradicts the connectedness of $[0, 1]$ and so our supposed path $\alpha$ cannot exist. Thus $X$ is not path connected.

\[\text{Figure 1. The path connected components of } V \text{ for the case of } \alpha(t_0) = (0, 0). \text{ The picture looks similar for larger values of } \alpha(t_0).\]

With our investigations of connectedness and path connectedness thus far, we can now partially answer question 3.35 about when Euclidean $n$-space $(\mathbb{R}^n, T_{Eu})$ and Euclidean $m$-space $(\mathbb{R}^m, T_{Eu})$ can be homeomorphic.
Corollary 7.26. If the Euclidean line \((\mathbb{R}, T_{Eu})\) is homeomorphic to Euclidean \(n\)-dimensional space \((\mathbb{R}^n, T_{Eu})\), then \(n = 1\).

Proof. Suppose that \(f : \mathbb{R} \rightarrow \mathbb{R}^n\) is a homeomorphism. Then \(f|_{\mathbb{R} - \{0\}} : \mathbb{R} - \{0\} \rightarrow \mathbb{R}^n - \{f(0)\}\) is also a homeomorphism. However, \(\mathbb{R} - \{0\}\) is not an interval and is therefore not connected according to theorem \(7.9\). On the other hand, we claim that \(\mathbb{R}^n - \{f(0)\}\) is even path connected, and therefore connected, when \(n \geq 2\). To see this, let \(x, y \in \mathbb{R}^n\) be any two points and consider the straight line path \(\alpha(t) = x + t(y - x)\). If \(f(0)\) does not lie on \(\alpha\), then \(\alpha\) is a path in \(\mathbb{R}^n - \{f(0)\}\) from \(x\) to \(y\). If \(f(0)\) does lie on \(\alpha\), let \(z \in \mathbb{R}^n\) be any point not collinear with \(x\) and \(y\) and consider the straight line paths \(\beta(t) = x + t(z - x)\) and \(\gamma(t) = z + t(y - z)\) connecting \(x\) to \(z\) and \(z\) to \(y\) respectively. Then \(\beta \cdot \gamma\) is a path from \(x\) to \(y\) in \(\mathbb{R}^n - \{f(0)\}\) showing that \(\mathbb{R}^n - \{f(0)\}\) is path connected. Since connectedness is a topological invariant, it follows that \(n\) must be 1.

In some instances, connectedness and path connectedness are in fact equivalent. Here is one class of such examples.

Theorem 7.27. An open subset \(U\) of Euclidean space \(\mathbb{R}^n\) is connected if and only if it is path connected.

Proof. In view of theorem \(7.22\), we only need to show that if \(U\) is connected then it is also path connected. Let \(x \in U\) be any point and define

\[
A = \{ y \in U \mid x \text{ and } y \text{ can be joined by a path in } U \} \\
B = \{ y \in U \mid x \text{ and } y \text{ can not be joined by a path in } U \}
\]

Clearly \(X = A \cup B\) and \(A \neq \emptyset\) since \(x \in A\). We will show that both \(A\) and \(B\) are open subsets of \(U\). Since \(U\) is connected, this will imply that \(B = \emptyset\) and \(A = U\), as desired.

To see that \(A\) is open, let \(y \in A\) be any point and let \(\alpha : [0, 1] \rightarrow U\) be a path joining \(x\) to \(y\). Let \(\varepsilon > 0\) be such that \(B_y(\varepsilon) \subset U\) and for any \(z \in B_y(\varepsilon)\) let \(\beta_z : [0, 1] \rightarrow U\) be the radial path from \(y\) to \(z\), i.e. \(\beta_z(t) = (1 - t) \cdot y + t \cdot z\). Then \(\alpha \cdot \beta_z\) is a path from \(x\) to \(z\) showing that \(B_y(\varepsilon) \subset A\). Since \(y \in A\) was arbitrary, we conclude that \(A\) is open.

To see that \(B\) is open, we proceed similarly. Let \(y \in B\) be any point and let \(\varepsilon > 0\) be such that \(B_y(\varepsilon) \subset U\). If there were a point \(z \in B_y(\varepsilon) \cap A\), there would have to be a path \(\alpha : [0, 1] \rightarrow U\) from \(z\) to \(x\). Letting \(\beta_z\) be as in the previous paragraph, the product path \(\beta_z \cdot \alpha\) would be a path from \(y\) to \(x\) contradicting our choice of \(y \in B\). Therefore \(B_y(\varepsilon)\) is contained in \(B\) and hence \(B\) is open.

Theorem 7.28. Let \((X, T_X)\) and \((Y, T_Y)\) be topological spaces and let \(X \times Y\) be given the product topology.

(a) If \(f : X \rightarrow Y\) is a continuous function and \(X\) is connected, then \(f(X)\) is a connected subspace of \(Y\). The same claim is true if “connectedness” is replaced by “path connectedness”.

(b) \(X \times Y\) is connected if and only if each of \(X\) and \(Y\) are connected. This claim too remain valid for path connectedness.
7.2. PATH CONNECTEDNESS

Proof. (a) If \( f(X) \) were not connected we could find two open, disjoint and non-empty subsets \( A, B \subset f(X) \) with \( f(X) = A \cup B \). But then \( f^{-1}(A) \) and \( f^{-1}(B) \) would also be two open (since \( f \) is continuous), non-empty and disjoint sets with \( X = f^{-1}(A) \cup f^{-1}(B) \). This would contradict the assumption that \( X \) is connected, therefore, \( f(X) \) must also be connected.

If \( X \) is path connected, pick two points \( f(x), f(y) \in f(X) \) and let \( \alpha : [0, 1] \to X \) be a path in \( X \) joining \( x \) to \( y \). Then \( f \circ \alpha : [0, 1] \to Y \) is path in \( Y \) joining \( f(x) \) to \( f(y) \) showing that \( Y \) is path connected.

(b) Since \( X \) and \( Y \) get surjected on by the projection maps \( \pi_X, \pi_Y \), part (a) of the theorem guarantees that \( X \) and \( Y \) are connected or path connected if \( X \times Y \) is connected or path connected.

In the other direction, let us assume that \( X \) and \( Y \) are connected. We will first show that for any points \( x_0 \in X \) and \( y_0 \in Y \), the subspaces
\[
X \times \{y_0\}, \quad \{x_0\} \times Y \quad \text{and} \quad Z = X \times \{y_0\} \cup \{x_0\} \times Y
\]
of \( X \times Y \) are connected. For the first two this is immediate and follows from corollaries 5.9 and 7.3. For the third one, suppose we could write it as a union \( A \cup B \) with \( A, B \) two open, disjoint and non-empty subsets of \( Z \). Then one of \( A \cap (X \times \{y_0\}) \) and \( B \cap (X \times \{y_0\}) \) has to be the empty set since they are open and disjoint subsets of the connected space \( X \times \{y_0\} \). For concreteness, let suppose that \( B \cap (X \times \{y_0\}) = \emptyset \) and note that then \( (x_0, y_0) \in A \). An analogous consideration forces one of \( A \cap \{x_0\} \times Y \) and \( B \cap \{x_0\} \times Y \) to be empty. If \( B \cap \{x_0\} \times Y \) were the empty set, then \( B \) would itself be the empty set, a contradiction. So we are led to conclude that \( A \cap \{x_0\} \times Y \) = \( \emptyset \), yet again a contradiction since this would force \( (x_0, y_0) \in A \cap B \). Thus \( Z \) must be connected.

It is now not hard to show that \( X \times Y \) is connected. Suppose we can obtain \( X \times Y \) as the union \( A \cup B \) with \( A, B \) open and disjoint subset of \( X \times Y \). By connectedness, each \( (X \times \{y_0\}) \cup \{x_0\} \times Y \) has to be contained entirely in either \( A \) or \( B \). The same is true of \( (X \times \{y_1\}) \cup \{x_0\} \times Y \) for every other \( y_1 \). But
\[
[(X \times \{y_0\}) \cup \{x_0\} \times Y] \cap [(X \times \{y_1\}) \cup \{x_0\} \times Y] = \{x_0\} \times Y
\]
so that both of \( (X \times \{y_0\}) \cup \{x_0\} \times Y \) and \( (X \times \{y_1\}) \cup \{x_0\} \times Y \) lie in the same set, either \( A \) or \( B \). Since \( y_0, y_1 \in Y \) were completely arbitrary, we see that all of \( (X \times \{y_0\}) \cup \{x_0\} \times Y \) lie in the same set and hence so does all of \( X \times Y \). Does one of the sets \( A \) or \( B \) has to be empty and so \( X \times Y \) is connected.

We still need to show that if \( X \) and \( Y \) are path connected then so is \( X \times Y \). This is easy, namely, given two points \( (x_1, y_1), (x_2, y_2) \in (X \times Y) \), let \( \alpha : [0, 1] \to X \) and \( \beta : [0, 1] \to Y \) be paths joining \( x_1 \) to \( x_2 \) and \( y_1 \) to \( y_2 \). Then \( \alpha \times \beta : [0, 1] \to X \times Y \) is a path joining \( (x_1, y_1) \) to \( (x_2, y_2) \).

The following lemma is the “path connected” version of lemma 7.14.

Lemma 7.29. Let \( (X, T_X) \) be a topological space and \( Y_i \subset X \), \( i \in I \) be a family of path connected subspaces. If \( \cap_{i \in I} Y_i \neq \emptyset \) then \( \cup_{i \in I} Y_i \) is a path connected subspace of \( X \).
Proof. Let \( p \in \bigcap_{i \in I} Y_i \) be any point and for \( x \in Y_i \), let \( \alpha_x \) be a path in \( Y_i \) connecting \( x \) to \( p \). Given any \( x, y \in \bigcup_{i \in I} Y_i \), the path \( \alpha_x \cdot \bar{\alpha}_y \) connects \( x \) to \( y \). \( \square \)

Definition 7.30. Let \((X, \mathcal{T}_X)\) be a topological space. A path connected component of \( X \) is any maximal path connected subspace \( U \) of \( X \). Maximality here refers to the property that if \( V \) is a path connected subspace of \( X \) with \( U \subset V \) then \( U = V \).

Example 7.31. The topologist's sine curve defined in example 2.18 and considered again in example 7.25 has two path connected components, namely \( Y_1 = \{(x, \sin \frac{1}{x}) | x \in (0, 1]\} \) and \( Y_2 = \{0\} \times [0, 1] \), but only has one connected component.

Here then is the analogue of theorem 7.15 for the path connected case.

Theorem 7.32. Let \((X, \mathcal{T}_X)\) be a topological space.
(a) \( X \) is the disjoint union of its path connected components.
(b) \( X \) is path connected if and only if it has a single path connected component.

Proof. (a) For a given \( x \in X \), the path connected component \( V_x \) containing it can be obtained as \( V_x = \bigcup_{V \in \mathcal{V}} V \) with \( \mathcal{V} = \{W \subset X | x \in W \text{ and } W \text{ is path connected}\} \).

The set \( V_x \) is path connected by the result of lemma 7.29 and it is clearly maximal with respect to that property. Two path connected components \( U, V \subset X \) are disjoint for if they were not, lemma 7.29 would imply that \( U \cup V \) is also path connected, contradicting the maximality of both \( U \) and \( V \) with respect to path connectedness.

(b) This proof is the complete analogue of the proof of part (c) of theorem 7.15 and is left as an exercise. \( \square \)

Unlike in the case of connectedness (see part (b) of theorem 7.15), a path connected component of \( X \) need not be closed nor open. Here is an example.

Example 7.33. Consider the topologist's sine curve \( X \) from example 2.18 (see also example 7.25). The two subsets \( Y_1, Y_2 \subset X \) defined as
\[
Y_1 = \{(x, \sin \frac{1}{x}) | x \in (0, 1]\} \quad Y_2 = \{0\} \times [0, 1]
\]
note that they are path connected seeing as they are homeomorphic to \((0, 1]\) and \([0, 1]\) respectively. Since \( X = Y_1 \cup Y_2 \) and \( X \) is not path connected by example 7.25, \( Y_1 \) and \( Y_2 \) are the only two path connected components of \( X \). It is easy to see that \( Y_2 \) is closed in \( X \) for \( Y_2 = X \cap (\{0\} \times \mathbb{R}) \). However, \( Y_2 \) is not open in \( X \) for if it were we could find an open set \( U \subset \mathbb{R}^2 \) with \( Y_2 = U \cap X \). But then, given any point \((0, y) \in Y_2 \), we would need to be able to find an \( \varepsilon > 0 \) with \( B_{(0,x)}(\varepsilon) \subset U \), an impossibility since each such \( B_{(0,y)}(\varepsilon) \) will intersect \( Y_1 \).

We conclude that

The path connected component \( Y_1 \) is open but not closed in \( X \).
The path connected component \( Y_2 \) is closed but not open in \( X \).
The reason for this deviation of behavior of the path connected components from the behavior of the connected components (part (b) of theorem 7.15) is the lack of an analogue of theorem 7.10 for the path connected case.

### 7.3. Local connectivity

**Definition 7.34.** Let \((X, T_X)\) be a topological space.

(a) We say that \(X\) is **locally connected** if every point \(x \in X\) has a neighborhood basis \(B_x\) consisting of connected open sets. Said differently, we require that for every point \(x \in X\) and every neighborhood \(U\) of \(x\) there exists a connected open set \(V \subset X\) with \(x \in V \subset U\).

(b) We say that \(X\) is **locally path connected** if every point \(x \in X\) has a neighborhood basis \(B_x\) consisting of path connected sets. Said differently, we require that for every point \(x \in X\) and every neighborhood \(U\) of \(x\) there exists an path connected open set \(V \subset X\) with \(x \in V \subset U\).

The next lemma is proved in complete analogy with theorem 7.22, we omit details.

**Lemma 7.35.** A locally path connected space is path connected.

**Example 7.36.** Every open subset \(U\) of Euclidean space \((\mathbb{R}^n, T_{Eu})\) is locally path connected and hence locally connected. This is obvious since, given any point \(x \in \mathbb{R}^n\) and given any neighborhood of \(U\) of \(x\), there exists an \(\varepsilon > 0\) with \(B_x(\varepsilon) \subset U\). Any such \(B_x(\varepsilon)\) is path connected by example 7.24.

The notions of path connectedness of \(X\) and of local path connectedness of \(X\) are completely independent in that neither implies the other. To verify this, here are two examples.

**Example 7.37 (A path connected but not locally path connected space).** Let \(X \subset \mathbb{R}^2\) be the infinite broom from example 2.19, i.e. let \(X\) be the subspace of the Euclidean plane given by

\[
X = \bigcup_{n=0}^{\infty} I_n \quad \text{with} \quad I_n = \begin{cases} 
\{ (t, t/n) \in \mathbb{R}^2 \mid t \in [0, 1] \} & ; \quad n \in \mathbb{N} \\
\{ (t, 0) \in \mathbb{R}^2 \mid t \in [0, 1] \} & ; \quad n = 0
\end{cases}
\]

It is easy to see that \(X\) is path connected (and hence also connected): If \(x \in X\) is any point then \(\alpha_x : [0, 1] \to X\), given by \(\alpha_x(t) = x - tx\), is a path connecting \(x\) to the origin. Given any two points \(x_1, x_2 \in X\), the path \(\alpha_{x_1} \cdot \overline{\alpha}_{x_2}\) connects the former to the latter. However, \(X\) is not locally connected and hence also not locally path connected. To see this, consider the point \(x_0 = (1, 0) \in X\) and let \(U \subset X\) be the neighborhood \(U = B_{y_0}(\frac{1}{2}) \cap X\) of \(x_0\) in \(X\). We claim that \(U\) contains no connected neighborhood of \(x_0\). For if \(V \subset U\) were a connected neighborhood of \(x_0\), we could find some \(\varepsilon > 0\) so that \(B_{x_0}(\varepsilon) \cap X \subset V\). But every such set has infinitely many path connected (and hence connected) components as is evident from figure 2.
Figure 2. The two neighborhoods $U$ and $B_{x_0}(\varepsilon)$ from example 7.37. The intersection of the infinite broom $X$ with $B_{x_0}(\varepsilon)$ is an infinite disjoint union of intervals and is therefore not connected.

Example 7.38 (A locally path connected but not path connected space). The subspace $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$ of the Euclidean line is both locally connected and locally path connected by not connected nor path connected.

Every discrete space $(X, \mathcal{T}_{\text{dis}})$ with at least two points is locally connected and locally path connected by not connected nor path connected.

Theorem 7.39. Let $(X, T_X)$ be a topological space.

(a) $X$ is locally connected if and only if every connected component of every open set $U \subset X$ is an open subset of $X$.

(b) $X$ is locally path connected if and only if every path connected component of $X$ is an open subset of $X$.

Proof. (a) $\implies$ Suppose that $X$ is locally connected, let $U \subset X$ be a an open subset and let $U_0 \subset U$ be a connected component of $U$. Then every point $x \in U_0$ has a connected neighborhood $U_{x,0}$. It is necessary that $U_{x,0}$ be contained in $U_0$ for if not, then $U_0 \cup U_{x,0}$ would be a connected subspace of $U$ (according to lemma 7.14), violating the maximality condition from the definition of a connected component (definition 7.13). Given this, we can write

$$U_0 = \bigcup_{x \in U} U_{x,0}$$

showing that $U_0$ is an open set.

$\impliedby$ Suppose the connected components of open subsets of $X$ are open sets. Let $x \in X$ be any point and $U$ any neighborhood of $x$. Then $U_x$, the connected component of $U$ that contains $x$, is a connected, open set with $x \in U_x \subset U$, showing that $X$ is locally connected.

(b) This is proved in complete analogy with part (a) by substituting “path connected” for “connected” in the proof of part (a) of the theorem. \hfill $\square$
The next corollary is easily deduced from theorems 7.15 and 7.39.

**Corollary 7.40.** In a locally connected space, the connected components are both open and closed. Similarly, in a locally path connected space, the path connected components are both open and closed.

**Proof.** The connected case is immediate from theorems 7.15 and 7.39.

For the path connected case, let $X$ be a path connected space and let $U_i, i \in \mathcal{I}$ be its path connected components. If $x \in U_i$ and $U_x$ is a path connected neighborhood of $x$, then $U_x \subset U_i$ by lemma 7.29. Thus $U_i = \bigcup_{x \in U_i} U_x$ showing that $U_i$ is open. But then $\bigcup_{j \in \mathcal{I} - \{i\}} U_j$ is also open and equal to $X - U_i$ showing that $U_i$ is closed. \qed

### 7.4. Exercises

7.1.

7.2.

7.3.