NOTES FOR DAYS 3, 4 and 5

Section 2.3 and Chapter 3: field and order axioms for the real numbers. (Chapter 3 is really a continuation of sec 2.3.)

On Sept 8 hand in problems 4 — 11 pages 53-54.

NOTE: We're going to consolidate the field axioms (**Properties 2.3.1**) on pages 18-19 along with three order properties. Eccles introduces four "order axioms" on page 24 in Chapter 3. We are going to do something slightly different. We are going to assume three much simpler order properties for the real numbers and derive the ones on page 24 from these. **See the notes for this.**

A *proof* is an argument that uses logical principles to conclude the truth of a statement from facts that are known or assumed true.

A *theorem* is a statement for which we have a proof. In mathematical writing, the term "theorem" is typically reserved for true statements whose proofs are not short or obvious. Other names for theorems, depending on the circumstances, are *lemma, proposition*, and *corollary.*

A *conjecture* is an assertion that we suspect may be true, but has not been proved.

Proof by Contradiction. Chapter 4

Suppose I want to prove that a statement p is true. A proof by contradiction goes like this: ***Assume*** p ***is false and deduce a contradiction – that is, deduce something that is known to be false***.

Here's what this means in logical symbols. Suppose r is a statement that I know is false. I prove $(\ NOT\ p) \Rightarrow r$ is true. Since r is a statement that I know is false I claim that this proves $(\ NOT\ p)$ must be false and therefore p must be true!

Example: Suppose I want to prove that the statement $p \Rightarrow q$ is true by contradiction. I assume it's false and try to arrive at a contradiction. But the negation of $p \Rightarrow q$ is logically equivalent to $(p\ AND\ \ NOT\ q)$. This means we are assuming p is true and q is false and from this we want to derive a third statement r that is false (this is the contradiction).

Summary: To prove that the statement $p \Rightarrow q$ is true by contradiction, ***assume*** p ***is true and*** q ***is false and deduce a contradiction – that is, deduce something that is known to be false***.

Exercise 17 *Proving* $p \Rightarrow q$ *by contradiction seems very much like proving the contrapositive of* $p \Rightarrow q$*. Explain the difference.*

Remark: Proofs by contradiction or contrapositive require you to negate a statement. This becomes more complicated when quantifiers are involved, as we will see in Chapter 7.

Exercise 18 *Suppose you are a high school math teacher and one of your students asks you why we can't divide by 0. How would you explain? (Saying it's illegal, or undefined, or against the rules is not an explanation. How would you explain why it's "illegal"?) Try this using the field axioms and again in a way that a high school kid might better understand.*

Sections 13.1, 13.2: Rational and irrational numbers.

Jump to Chapter 13, sec 13.1 & 13.2 only. Save the rest of Chapter 13 for later.

If a and b are real numbers with $a \neq 0$ then by the field axioms, there is a unique number x such that $ax = b$. ($x = a^{-1}b$ which is also written as $x = \frac{b}{a}$.)

Definition: x is a rational number if and only if there are integers a and b with $a \neq 0$ such that $ax = b$. A number is called *irrational* if and only if it is not rational.

Exercises:

1. If x and y are rational numbers, so is $x + y$.
2. If x and y are rational numbers, so is xy.
3. If x and y are rational numbers and $x \neq 0$, then $\frac{y}{x}$ is rational.
4. If x is irrational and $y \neq 0$ is rational, then xy is irrational.
5. If x is irrational and y is rational, then $x + y$ is irrational.
6. Prove or give a counterexample: If x is irrational and y is irrational, then $x+y$ is irrational.
7. Prove or give a counterexample: If x is irrational and y is irrational, then xy is irrational.

[illegible]
Notes for
field + order
Ch 2.2

The Order Properties of $\mathbb{R}$

The "order properties" of $\mathbb{R}$ refer to the notions of positivity and inequalities between real numbers. As with the algebraic structure of the system of real numbers, we proceed by isolating three basic properties from which all other order properties and calculations with inequalities can be deduced. The simplest way to do this is to identify a special subset of $\mathbb{R}$ by using the notion of "positivity".

2.1.5 The Order Properties of $\mathbb{R}$ There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of **positive real numbers**, that satisfies the following properties:

(i) If a, b belong to $\mathbb{P}$, then $a + b$ belongs to $\mathbb{P}$. $= \mathbb{R}^+$ in Eccles

(ii) If a, b belong to $\mathbb{P}$, then ab belongs to $\mathbb{P}$.

(iii) If a belongs to $\mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

The first two conditions ensure the compatibility of order with the operations of addition and multiplication, respectively. Condition 2.1.5(iii) is usually called the **Trichotomy Property**, since it divides $\mathbb{R}$ into three distinct types of elements. It states that the set $\{-a : a \in P\}$ of **negative** real numbers has no elements in common with the set $\mathbb{P}$ of positive real numbers, and, moreover, the set $\mathbb{R}$ is the union of three disjoint sets.

If $a \in \mathbb{P}$, we write $a > 0$ and say that a is a **positive** (or a **strictly positive**) real number. If $a \in \mathbb{P} \cup \{0\}$, we write $a \geq 0$ and say that a is a **nonnegative** real number. Similarly, if $-a \in \mathbb{P}$, we write $a < 0$ and say that a is a **negative** (or a **strictly negative**) real number. If $-a \in \mathbb{P} \cup \{0\}$, we write $a \leq 0$ and say that a is a **nonpositive** real number.

The notion of inequality between two real numbers will now be defined in terms of the set $\mathbb{P}$ of positive elements.

2.1.6 Definition Let a, b be elements of $\mathbb{R}$.

(a) If $a - b \in \mathbb{P}$, then we write $a > b$ or $b < a$.

(b) If $a - b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

The Trichotomy Property 2.1.5(iii) implies that for $a, b \in \mathbb{R}$ exactly one of the following will hold:

$$a > b, \qquad a = b, \qquad a < b.$$

Therefore, if both $a \leq b$ and $b \leq a$, then $a = b$.

For notational convenience, we will write

$$a < b < c$$

to mean that both $a < b$ and $b < c$ are satisfied. The other "double" inequalities $a \leq b < c$, $a \leq b \leq c$, and $a < b \leq c$ are defined in a similar manner.

To illustrate how the basic Order Properties are used to derive the "rules of inequalities", we will now establish several results that the reader has used in earlier mathematics courses.

2.1.7 Theorem *Let a, b, c be any elements of $\mathbb{R}$.*

(a) *If $a > b$ and $b > c$, then $a > c$.*

(b) *If $a > b$, then $a + c > b + c$.*

(c) *If $a > b$ and $c > 0$, then $ca > cb$.*
If $a > b$ and $c < 0$, then $ca < cb$.

Proof. (a) If $a - b \in \mathbb{P}$ and $b - c \in \mathbb{P}$, then 2.1.5(i) implies that $(a - b) + (b - c) = a - c$ belongs to $\mathbb{P}$. Hence $a > c$.
(b) If $a - b \in \mathbb{P}$, then $(a + c) - (b + c) = a - b$ is in $\mathbb{P}$. Thus $a + c > b + c$.
(c) If $a - b \in \mathbb{P}$ and $c \in \mathbb{P}$, then $ca - cb = c(a - b)$ is in $\mathbb{P}$ by 2.1.5(ii). Thus $ca > cb$ when $c > 0$.

On the other hand, if $c < 0$, then $-c \in \mathbb{P}$, so that $cb - ca = (-c)(a - b)$ is in $\mathbb{P}$. Thus $cb > ca$ when $c < 0$. Q.E.D.

It is natural to expect that the natural numbers are positive real numbers. This property is derived from the basic properties of order. The key observation is that the square of any nonzero real number is positive.

The product of two positive numbers is positive. However, the positivity of a product of two numbers does not imply that each factor is positive. The correct conclusion is given in the next theorem. It is an important tool in working with inequalities.

2.1.10 Theorem *If $ab > 0$, then either*

(i) *$a > 0$ and $b > 0$, or*

(ii) *$a < 0$ and $b < 0$.*

Proof. First we note that $ab > 0$ implies that $a \neq 0$ and $b \neq 0$. (Why?) From the Trichotomy Property, either $a > 0$ or $a < 0$. If $a > 0$, then $1/a > 0$ (why?), and therefore $b = (1/a)(ab) > 0$. Similarly, if $a < 0$, then $1/a < 0$, so that $b = (1/a)(ab) < 0$. Q.E.D.

2.1.11 Corollary *If $ab < 0$, then either*

(i) *$a < 0$ and $b > 0$, or*

(ii) *$a > 0$ and $b < 0$.*

Exercises for Section 2.1

1. If $a, b \in \mathbb{R}$, prove the following.
   (a) If $a + b = 0$, then $b = -a$,
   (b) $-(-a) = a$,
   (c) $(-1)a = -a$,
   (d) $(-1)(-1) = 1$.
2. Prove that if $a, b \in \mathbb{R}$, then
   (a) $-(a + b) = (-a) + (-b)$,
   (b) $(-a) \cdot (-b) = a \cdot b$,
   (c) $1/(-a) = -(1/a)$,
   (d) $-(a/b) = (-a)/b$ if $b \neq 0$.
3. Solve the following equations, justifying each step by referring to an appropriate field axiom.
   (a) $2x + 5 = 8$,
   (b) $x^2 = 2x$,
   (c) $x^2 - 1 = 3$,
   (d) $(x - 1)(x + 2) = 0$.
4. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.
5. If $a \neq 0$ and $b \neq 0$, show that $1/(ab) = (1/a)(1/b)$.

10. (a) If $a < b$ and $c \leq d$, prove that $a + c < b + d$.
    (b) If $0 < a < b$ and $0 \leq c \leq d$, prove that $0 \leq ac \leq bd$.

11. (a) Show that if $a > 0$, then $1/a > 0$.
    (b) Show that if $a < b$, then $a < \frac{1}{2}(a + b) < b$.

11(c) $0 < x < y \Rightarrow \frac{1}{y} < \frac{1}{x}$

Def: $2 = 1+1$.

12. Let a, b, c, d be numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, and one where $bd < ac$.

13. If $a, b \in \mathbb{R}$, show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$.

14. If $0 \leq a < b$, show that $a^2 \leq ab < b^2$. Show by example that it does *not* follow that $a^2 < ab < b^2$.

15. If $0 < a < b$, show that (a) $a < \sqrt{ab} < b$, and (b) $1/b < 1/a$.

16. Find all real numbers x that satisfy the following inequalities.
    (a) $x^2 > 3x + 4$,
    (b) $1 < x^2 < 4$,
    (c) $1/x < x$,
    (d) $1/x < x^2$.

17. Prove If $a \in \mathbb{R}$ is such that $0 \leq a \leq \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

18. Let $a, b \in \mathbb{R}$, and suppose that for every $\varepsilon > 0$ we have $a \leq b + \varepsilon$. Show that $a \leq b$.

19. Prove that $\left[\frac{1}{2}(a + b)\right]^2 \leq \frac{1}{2}\left(a^2 + b^2\right)$ for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

20. (a) If $0 < c < 1$, show that $0 < c^2 < c < 1$.
    (b) If $1 < c$, show that $1 < c < c^2$.