NOTES FOR DAYS 6 & 7

Proof by Mathematical Induction. Chapter 5. DAY 8 Exam 1

Let $P(n)$ be a proposition about the natural number n. **Mathematical Induction** says the following:

If $P(1)$ Is true and if $[P(n) \Rightarrow P(n+1)]$ is true for all natural numbers n then $P(n)$ is true for every natural number n.

Be aware that mathematical induction is actually an *axiom* of the natural numbers.

The following is called the **Principle of Strong Induction.** (Sometimes ordinary induction is referred to as " weak induction".)

If $P(1)$ is true if $[(P(1)\ AND\ \ldots\ AND\ P(n)) \Rightarrow P(n+1)]$ is true for all natural numbers n then $P(n)$ is true for every natural number n.

Strong induction follows from mathematical induction. **Here's the proof.**

Let $Q(n)$ be the statement $P(1)\ AND\ \ldots\ AND\ P(n)$.

Then we want to show that if $P(1)$ is true and $[Q(n) \Rightarrow P(n+1)]$ is true for all natural numbers n then $P(n)$ is true for every natural number n.

What we're going to do is use mathematical induction to show that $Q(n)$ is true for every natural number n. Since $Q(n) \Rightarrow P(n)$ it will follow that $P(n)$ is true for every natural number n.

First, notice that $Q(1)$ is the same as $P(1)$. Therefore, since $P(1)$ is assumed to be true, $Q(1)$ is true.

Also for each natural number n, $[Q(n) \Rightarrow P(n+1)]$. Thus for each natural number n, $[Q(n) \Rightarrow (Q(n)\ AND\ P(n+1)) \iff Q(n+1)]$.

Since $Q(1)$ is true and $[Q(n) \Rightarrow Q(n+1)]$ is true for each natural number n, it follows that $Q(n)$ is true for all n by induction. QED.

Exercise 19 *Prove these interesting facts using mathematical induction.*

1. For all natural numbers n, $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
2. For all natural numbers n, $\displaystyle\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$

3. Suppose $a \neq 1$ and $a \neq 0$. For all natural numbers n, $\sum_{i=1}^{n} a^{i-1} = \frac{1-a^n}{1-a}$

4. If a is a real number with $a > -1$, then for all natural numbers k, $(1+a)^k \geq 1 + ak$

5. For all natural numbers k, $9^k - 8k - 1$ is divisible by 64.

6. $5^n - 4n - 1$ is divisible by 8 for every natural number n.

7. Show that if $2 + 4 + 6 + ... + 2n = n(n+1) + 2$ is true for $n = n_0$, then it is true for $n = n_0 + 1$. Is the formula true for all n?

8. The set of natural numbers (denoted **N**) is what is called a "well ordered set". What this means is that every nonempty subset of **N** has any smallest element. Use the principle of strong induction to prove this.

   Hint: Suppose S is a nonempty set of natural numbers with no least element, and let T be the set of numbers in **N** that are not in S. Use strong induction to prove that T = **N**. This contradicts the fact that the set S is nonempty.

9. Try to prove the above theorem using only "weak" induction, and not strong induction.

22. (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$, and that $c^n > c$ for $n > 1$.
    (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, and that $c^n < c$ for $n > 1$.

23. If $a > 0, b > 0$ and $n \in \mathbb{N}$, show that $a < b$ if and only if $a^n < b^n$. [Hint: Use Mathematic Induction].

24. (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.
    (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m < c^n$ if and only if $m > n$.

25. Assuming the existence of roots, show that if $c > 1$, then $c^{1/m} < c^{1/n}$ if and only if $m > n$.

26. Use Mathematical Induction to show that if $a \in \mathbb{R}$ and $m, n, \in \mathbb{N}$, then $a^{m+n} = a^m a^n$ an $(a^m)^n = a^{mn}$.