NOTES ON LOGIC AND PROOFS. DAYS 1 and 2

Chapters 1 and 2 through Sec 2.2.

On Sept 1 hand in Problems 1 & 2 page 53 of Eccles plus # 1-13 in these notes.

We will assume that variables always stand for real numbers unless specifically stated otherwise.

The author of your text makes a distinction between the term "proposition" and the term "statement". In his vocabulary a *proposition* is a sentence that is either true or false (but not both). For example, $x^2 = 4$ is not a proposition because it may be true or false depending on the value of x.

Likewise, the sentence, "He is short." is not a proposition either because it is ambiguous. Some people might call a 5 foot 7 inch man short while others would say that he's not really short, he's just sort of medium. Such ambiguity is not acceptable in mathematics.

Which of these is a proposition in the above sense? (a) $25 = 1$ (b) $x^3 = 0$ (c) $7 \geq 1$ (d) $\sqrt{2}$ (e) $x + 1 = 1 + x$ (f) "This sentence is false."

A *predicate* (see p. 4) is a sentence that contains variables (which the author calls *free variables*), and may be true or false depending on the assignment of values to those variables.

Which of these is a predicate? (a) $25 = 1$ (b) $x^2 = 4$ (c) $x + 1 = 1 + x$ (d) $7 \geq 1$ (e) $\sqrt{2}$ (f) "This sentence is false."

A *statement* (see p. 4) is either a proposition or predicate.

Which of these is a statement? (a) $25 = 1$ (b) $x^2 = 4$ (c) $x + 1 = 1 + x$ (d) $7 \geq 1$ (e) $\sqrt{2}$ (f) "This sentence is false."

A statement like "the universe is infinite" is assumed to be either true or false, but scientists just don't know the answer. There are lots of unsolved problems in mathematics, i.e., statements that may be either true or false, but we don't know which. An example is given in your book called the Goldbach conjecture, which says that every even integer greater than two is the sum of two primes.

Remark: In 1931 the mathematician Kurt Gödel proved roughly the following theorem: any consistent set of axioms from which the ordinary rules of arithmetic can be derived must be incomplete, that is, there must be statements that cannot be proved true and cannot be proved false. Therefore, it's entirely possible that the Goldbach conjecture is one of these "undecidable" statements.

Keeping in mind that statements are assumed to be either true or false, is this statement true? *"All the elephants in my shirt pocket are wearing ballet slippers."* Why or why not?

OR and AND

There is no question as to what "and" means, but this is not the case with "or".

In ordinary conversation, if I say, "I will take you to the 3 o'clock showing of Harry Potter or the 7 o'clock showing," I mean one or the other but not both. This is an illustration of the so-called "exclusive or." On the other hand, if a waitress says, "Would you like sugar or cream in your coffee?" she means one or the other or possibly both. This is an illustration of the so-called "inclusive or."

In mathematics we always use the word "or" in the *inclusive* sense − one or the other or possibly both. Thus all three of the following statements are true

$$(2+2=4)\ OR\ (3+3=5)\ , \quad (2+2=5)\ OR\ (3+3=6)\ , \quad (2+2=4)\ OR\ (3+3=6)\ .$$

The only "or" statement that is false is one with both component parts false, as for example $(2+2=5)\ OR\ (3+3=5)$.

p	q	$p\ AND\ q$	$p\ OR\ q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

NOT

The *negation* of a statement p is sometimes written $NOT\ p$ and it means "it is not the case that p". $NOT\ p$ is true when, and only when p is false.

p	q	$NOT\ q$	$p\ AND\ (\ NOT\ q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

PARENTHESES.

What does this mean? $3 < 1\ AND\ 2 = 2\ OR\ 5 = 5$. Does it mean $(3 < 1\ AND\ 2 = 2)\ OR\ 5 = 5$ or does it mean $3 < 1\ AND\ (2 = 1\ OR\ 5 = 5)$? The first is true but the second is false.

The point is that parentheses are important. You must use them to avoid ambiguity.

Another example is $NOT\ p\ AND\ q$. It could mean $NOT\ (p\ AND\ q)$ or $(\ NOT\ p)\ AND\ q$. These are not the same. Usually $NOT\ p\ AND\ q$ is interpreted as $(\ NOT\ p)\ AND\ q$ just like $-2+3$ means $(-2)+3=1$ and not $-(2+3)=-5$. However, using parentheses ensures no confusion.

Exercise 1 *The negation of* $p\ OR\ q$ *is* $NOT\ (p\ OR\ q)$. *Write this in another way.*

Exercise 2 *Write the negation of this:* $(2+2=4)\ OR\ (3+3<6)$ *without using the word "not".*

Exercise 3 *The negation of* $p\ AND\ q$ *is* $NOT\ (p\ AND\ q)$. *Write this in another way.*

Exercise 4 *Write the negation of this:* $(2+2=4)\ AND\ (3+3<6)$ *without using the word "not".*

Exercise 5 *Write the negation of this: "All the elephants in my shirt pocket are wearing ballet slippers."*

IF ... THEN

A sentence of the form "if p then q" is called a *conditional*. The sentence following the word "if" is called the **hypothesis** or the **antecedent**. The part following "then" is called the **conclusion** or **consequent**. Here is its truth table:

p	q	If p then q
T	T	T
T	F	F
F	T	T
F	F	T

Few words are so misused and misunderstood as "if ... then", but it is absolutely vital that you understand clearly the meaning of these words as we will use them.

Think of a statement of the form "if p then q", as a promise with a condition (in fact, this is why such statements are called *conditionals*). I promise that **if** p happens then q will happen too. The promised q need not be fulfilled unless the condition p is met.

If you think about what it takes to make this statement false, there's only one way - that's when p is true but q is false. Otherwise you really can't say the promise has been broken. And if you can't say it's false in the other cases, then there's only one other choice - it's got to be true, because every statement in mathematics is either true or false.

Suppose I promise that you will get an "A" in the course if you have an average that is 90% or greater. If I am telling you the truth, then you will certainly expect an "A" if your average is 92%. But if your average is 89%, I can give you an "A" anyway without breaking my promise. I could also give you a "B" without breaking my promise. The question of what will occur for an average that is less than 90% is simply not addressed by the promise as stated, using those special words "if" and "then".

> You may use the notation $p \Rightarrow q$ to stand for "if p then q", but you must use it correctly. One often reads $p \Rightarrow q$ as "p implies q". **Remark: Eccles uses "$\Leftarrow$" a lot. This is uncommon and I ask that you avoid it.**

Exercise 6 *Which of these are true? (1) If* $2+2=5$ *then* $1 \leq 2$*; (2) If* $2+2<5$ *then 3 is an even number.*

Exercise 7 *Is it correct to write "if* $x=5$ *then* $x+2=7$*" as "if* $(x=5) \Rightarrow (x+2=7)$*"?*

"p if and only if q" is called a *biconditional*. It expresses the idea that the two statements p and q have the same truth value. Therefore the statement is true when, and only when, p and q are both true or both false.

> You may use the notation $p \iff q$ to stand for " p if and only if q", but you must use it correctly.

Thus the statement "$2+2=4$ if and only if $9=3$ times 3". is true. Never mind that you see no rhyme or reason for putting the two parts together. When two sentences are combined by using the words "if and only if", the result is true when the two are both true or when they are both false.

What about the following sentence? Is it true or false?

$7 < 5 \iff$ Garth Brooks is Harry Potter.

Both component parts are false, therefore the entire statement is true.

The sentence $(2+2=4) \iff (1=0)$ is false because the parts do not have the same truth value.

Logically equivalent statements.

Two statements are called <u>*logically equivalent*</u> if one is true when, *and only when,* the other is true. Another way to say this is that the truth tables of the two statements are identical.

Eccles addresses this on page 15 section 2.1 (i). Here he uses the term "equivalent" instead of "logically equivalent".

People sometimes use the terms "equal" and "equivalent" to mean the same thing. In mathematics, they have different meanings.

Exercise 8 *Which of these make sense? Which are true?*
(a) $2+2 \Rightarrow 4$. *(b)* $(x+2=3)=(x=1)$. *(c)* $(x+2=3) \Rightarrow (x=1)$
(d) $x=1 \Rightarrow x+2=3$ *(e)* $x=1 \Rightarrow x^2=1$ *(f)* $x^2=1 \Rightarrow x=1$.

It should be clear that p and q are logically equivalent exactly when $p \iff q$ is a true statement.

Here are some important examples:

(I) (a) $p \Rightarrow q$ and (b) q *OR* (*NOT* p) are logically equivalent. You can show this by a truth table or by observing that they are both false in exactly the same single case.

Exercise 9 *Write statements (1) If* $2+2=5$ *then* $1 \leq 2$*; (2) If* $2+2<5$ *then 3 is an even number in the form of (b) above* without *NOT* or the symbol $\Rightarrow$.

(II) (a) *NOT* $(p \Rightarrow q)$ and (b) p *AND* (*NOT* q) are logically equivalent. You can show this by a truth table or by observing that they are both true in exactly the same single case.

Exercise 10 *Write the negation of statements (1) If* $2+2=5$ *then* $1 \leq 2$*; (2) If* $2+2<5$ *then 3 is an even number in the form of (b) above* without *NOT* or the symbol $\Rightarrow$.

(III) The statement *NOT* $q \Rightarrow$ *NOT* p is called the <u>*contrapositive*</u> of $p \Rightarrow q$. (This is not mentioned in Eccles until Chapter 4, p. 34.) They are logically equivalent. You can show this by a truth table or by observing that they are both false in exactly the same single case. *It is often a very useful strategy to prove the contrapositive of a statement rather than the statement itself.*

See exercise 2.5 in Eccles.

Exercise 11 *Suppose that n is a natural number. Prove that if n^2 is even then n is even. It is not clear at all how one would prove this directly, but proving the contrapositive is not hard. State the contrapositive and prove it.*

Exercise 12 *Write the contrapositives of these statements* without *NOT* or the symbol $\Rightarrow$. *(1) If* $2+2=5$ *then* $1 \leq 2$. *(2) If* $2+2<5$ *then 3 is an even number.*

(IV) The statement $q \Rightarrow p$ is called the <u>*converse*</u> of $p \Rightarrow q$. They are <u>NOT</u> logically equivalent. You can show this by a truth table or by observing that when p is false and q is true they they have different truth values.

Exercise 13 *Write the converses of these statements* without *NOT* or the symbol $\Rightarrow$ *(1) If* $2+2=5$ *then* $1 \leq 2$*; (2) If* $2+2<5$ *then 3 is an even number.*

Exercise 14 *I am going to prove that* $1=-1$. *Here it is: Starting with* $1=-1$ *we square both sides to get* $1^2=(-1)^2$, *or* $1=1$, *which is true. This ends my proof. Is my proof correct?*

Exercise 15 *I am going to prove that* $x=-2$ *and* $x=1$ *are solutions to the equation* $\dfrac{x^2+x-2}{x+2}=0$. *Here it is: To solve* $\dfrac{x^2+x-2}{x+2}=0$, *multiply by* $x-2$ *to get* $x^2+x-2=0$. *Factor to get* $(x+2)(x-1)=0$. *Therefore,* $x=-2$ *and* $x=1$ *are both solutions. This ends my proof. Is my proof correct?*

Exercise 16 *I will solve* $x+3=\sqrt{2x+14}$. *Square both sides to get* $(x+3)^2=2x+14$, *which simplifies to* $x^2+6x+9=2x+14$ *or* $x^2+4x-5=0$. *Factor to get* $(x+5)(x-1)=0$, *which gives the solutions:* $x=1$ *and* $x=-5$. *This ends my proof. Is my proof correct?*