Proof of 5.26 (a): Let $x \in B \cup \bigcap_{j \in J} A_j$. Then $x \in B$ or $x \in \bigcap_{j \in J} A_j$.

If $x \in B$ then $x \in B \cup A_j$ for all j and thus $x \in \bigcap_{j \in J} B \cup A_j$.

On the other hand, if $x \in \bigcap_{j \in J} A_j$, then $x \in A_j$ for every j and hence $x \in B \cup A_j$ for every j. This means that $x \in \bigcap_{j \in J} B \cup A_j$.

In either case $x \in \bigcap_{j \in J} B \cup A_j$, so $B \cup \bigcap_{j \in J} A_j \subseteq \bigcap_{j \in J} B \cup A_j$.

Now, let $x \in \bigcap_{j \in J} B \cup A_j$. Then for each j, $x \in A_j$ or $x \in B$.

We want to show that $x \in B \cup \bigcap_{j \in J} A_j$, that is, either $x \in B$ or $\forall\, j \in J, x \in A_j$.

For any j, $x \in A_j$ or $x \in B$, so if $x \notin B$ then, $x \in A_j$. But this is true for all j, so $x \in \bigcap_{j \in J} A_j$.

Therefore, $\bigcap_{j \in J} B \cup A_j \subseteq B \cup \bigcap_{j \in J} A_j$

QED

NOTES:

In the second half of the proof, **we know** that for each j, x is in one of the two sets A_j or B. What we want **to show** is that either $x \in B$ or $x \in A_j$ for every j. This a subtle difference in wording.

Also, at the end we wanted to show that $x \in B$ or $\forall\, j \in J, x \in A_j$. This is an "OR" statement. To prove $p\ OR\ q$ we can prove the logically equivalent form $(\ NOT\ p) \Rightarrow q$. In this case that's $x \notin B \Rightarrow x \in A_j \forall\, j \in J$.

If $J = \{1, 2\}$, then $\bigcap_{j \in J} A_j = A_1 \cap A_2$. Thus, the above theorem says

$B \cup \bigcap_{j \in J} A_j = (B \cup A_1) \cap (B \cup A_2)$