Chapter 8. Functions and sequences.

Sections 8.1 and 8.2 of Eccles are about functions. This is very important because Calculus is really the study of functions and Intro Analysis (Math 310 & 311) is all about proving the theorems of calculus.

Note: In definition 8.1.1 on page 89, Eccles uses the terminology "co-domain". This is not common terminology.

Exercise 24 *Find the domains of these. Write them in interval notation except for (d). Remember a domain is a set, so express it as such. (No proofs needed.) All numbers are real except in the last one.*

a. $f(x) = \sqrt{1-2x}$.

b. $g(x) = \dfrac{3}{\sqrt{x^2-1}}$.

c. $K(x) = \dfrac{x}{\sqrt{1-x^2}}$.

d. $h : \mathbf{Z} \to \mathbf{R}$ by $h(x) = \sqrt{4-x^2}$.

Exercise 25 *Find and simplify:* $(f \circ g)(x)$, $(f \circ f)(x)$, $(g \circ f)(x)$, *and* $(g \circ g)(x)$.

a. $f(x) = 1 - 5x$; $g(x) = 3x + 1$

b. $f(x) = x^2$; $g(x) = 3x + 1$

c. $f(x) = x^2 + 1$; $g(x) = \sqrt{2x+1}$.

Section 8.3 is on sequences. We will dwell on this at some length. Eccles does not use conventional terminology here. Instead of (a_n) or $\{a_n\}$, which are used in most calculus books, Eccles writes $n \mapsto a_n$. This serves to emphasize that the sequence is a function whose domain is the set of natural numbers. However, it is uncommon notation. We will use the more conventional notation of $\{a_n\}$ instead of $n \mapsto a_n$

Eccles defines what he calls a "null sequence" as follows: $\{a_n\}$ is a ***null sequence*** if and only if $\forall \epsilon > 0\ \exists n \in \mathbb{N} \ni$ if $n \geq N$ then $|a_n| < \epsilon$.

We extend his definition as follows: $\{a_n\}$ ***converges*** to a number a if and only if $\{a_n - a\}$ is a null sequence. That is, $\forall \epsilon > 0 \; \exists n \in \mathbb{N} \ni$ if $n \geq N$ then $|a_n - a| < \epsilon$.

In the terminology of this definition, we may say that $\{a_n\}$ is a null sequence if and only if it converges to zero.

Exercise 26 *Suppose $\{a_n\}$ converges to a and that c is a non-zero real number. Prove that $\{ca_n\}$ converges to ca. (What if $c = 0$?)*

Exercise 27 *Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. Prove that $\{a_n + b_n\}$ converges to $a + b$.*

The Archimedean Property: Given any real number, there is a positive integer greater than that number. In symbols, it says $\forall x \in \mathbb{R} \; \exists n \in \mathbb{N} \ni n > x$. (See Eccles p. 166).

This property of the real numbers that may seem obvious to you, but it does not follow from the "field axioms", so how do you prove it? It actually has to be assumed or proved from additional axioms.

Eccles puts off mentioning the Archimedean property until page 166, but the following exercises as well as the proof given on page 97 of Eccles actually require it.

Example: Prove that the sequence $\left\{\frac{1}{n}\right\}$ converges to zero. Proof: Let $\epsilon > 0$ be given. By the Archimedean property, there is a natural number N such that $N > \frac{1}{\epsilon}$.

Let $n \in N$ with $n > N$. Then $n > \frac{1}{\epsilon}$, so $\frac{1}{n} < \epsilon$. this completes the proof.

Note the structure of the proof: (1) Let ϵ be given. (2) Find N. (3) Show N "works".

Exercise 28 *Prove the following.*

a. $\left\{\frac{n}{n^2+1}\right\}$ is a null sequence (i.e., it converges to 0).

b. $\left\{\frac{2n}{n+1}\right\}$ converges to 2.

c. $\left\{\frac{3n+1}{2n+5}\right\}$ converges to $\frac{3}{2}$.

d. $\left\{\frac{n^2-1}{2n^2+3}\right\}$ converges to $\frac{1}{2}$.

e. $\left\{\frac{1}{\sqrt{n}+7}\right\}$ is a null sequence (i.e., it converges to 0).

f. $\left\{\frac{\sqrt{n}}{n+1}\right\}$ is a null sequence (i.e., it converges to 0).