Logical Quantifiers. Chapter 7

$\forall$ stands for "for every" or "for all." $\exists$ stands for "there exists."

We often use the phrase "such that" with "there exists." Many authors use $\ni$ as shorthand for "such that." Others use a colon (:) for "such that".

For example, the sentence, "There exists a positive number" may be written as "There exists a number x such that $x > 0$", which may also be written as "$\exists x \ni x > 0$ or as "$\exists x : x > 0$.

NOTE: Eccles just puts a comma where one would use "such that." He would say $\exists x, x > 0$. This is unusual. I prefer that we use $\ni$ or a colon.

You may use the symbols $\forall$, $\exists$ and $\ni$, but you must use them correctly. They are often misused, so if you are not sure just write out the words.

NOTE: Later in the book Eccles defines a "null sequence" (which means that it converges to zero). It uses multiple quantifiers. In Math 310 you'll encounter lots of definitions like it and negating them will be useful.

Examples (Unless otherwise specified, assume x and y are real numbers.).

(a) The square of any real number is non-negative: $\forall x,\ x^2 \geq 0$. If it were not clear that x is a real number you could write: $\forall$ real number x, $x^2 \geq 0$.

(b) Every non-negative real number has a square root: $\forall x \geq 0, \exists y \ni y^2 = x$. Here's another way: $\forall x, (x \geq 0 \Rightarrow \exists y \ni y^2 = x)$.

(c) If $x + 1 = y + 1$, then $x = y$: $\forall x\ \forall y, (x + 1 = y + 1 \Rightarrow x = y)$. Here's another way: $\forall x, y$, $(x + 1 = y + 1 \Rightarrow x = y)$.

Exercise 22 *Express the negation of the statements (a), (b) and (c) above in words and using $\forall$ and $\exists$. (It is not acceptable to just write "not" or "it is not the case that" in front of the statement.) Which of the six statements are true?*

Exercise 23 *Prove that there are irrational numbers a and b such that a^b is rational. Hint: Either $\sqrt{2}^{\sqrt{2}}$ is irrational or rational. If it is irrational, then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. The proof here presents a very interesting situation, which we will discuss.*

1 **PRACTICE** Rewrite each statement using $\exists$, $\forall$, and $\ni$, as appropriate.

(a) There exists a positive number x such that $x^2 = 5$.
(b) For every positive number M there is a positive number N such that $N < 1/M$.
(c) If $n \geq N$, then $|f_n(x) - f(x)| \leq 3$ for all x in A.
(d) No positive number x satisfies the equation $f(x) = 5$.

2 **PRACTICE** Write the negation of each statement in Practice 1

EXERCISES

2.1 Mark each statement True or False. Justify each answer.

(a) The symbol "$\forall$" means "for every."
(b) The negation of a universal statement is another universal statement.
(c) The symbol "$\ni$" is read "such that."

2.2 Mark each statement True or False. Justify each answer.

(a) The symbol "$\exists$" means "there exist several."
(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.
(c) The order in which quantifiers are used affects the truth value.

2.3 Write the negation of each statement.

(a) All the roads in Yellowstone are open.
(b) Some fish are green.
(c) No even integer is prime.
(d) $\exists\, x < 3 \ni x^2 \geq 10$.
(e) $\forall\, x$ in A, $\exists\, y < k \ni 0 < f(y) < f(x)$.
(f) If $n > N$, then $\forall\, x$ in S, $|f_n(x) - f(x)| < \varepsilon$.

2.4 Write the negation of each statement.

(a) Some basketball players at Central High are short.
(b) All of the lights are on.
(c) No bounded interval contains infinite many integers.
(d) $\exists\, x$ in $S \ni x \geq 5$.
(e) $\forall\, x \ni 0 < x < 1$, $f(x) < 2$ or $f(x) > 5$.
(f) If $x > 5$, then $\exists\, y > 0 \ni x^2 > 25 + y$.

I. Which of the following are true? The domain for each is given in parentheses.

(a) $\forall x(x+1 \geq x)$ (Real numbers)

(b) $\exists x(2x+3=5x+1)$ (Natural numbers)

(c) $\exists x(x^2+1=2^x)$ (Real numbers)

(d) $\exists x(x^2=2)$ (Rational numbers)

(e) $\exists x(x^2=2)$ (Real numbers)

(f) $\forall x(x^3+17x^2+6x+100 \geq 0)$ (Real numbers)

(g) $\exists x(x^3+x^2+x+1 \geq 0)$ (Real numbers)

(h) $\forall x \exists y(x+y=0)$ (Real numbers)

(i) $\exists x \forall y(x+y=0)$ (Real numbers)

(j) $\forall x \exists! y(y=x^2)$ (Real numbers)

(k) $\forall x \exists! y(y=x^2)$ (Natural numbers)

(l) $\forall x \exists y \forall z(xy=xz)$ (Real numbers)

(m) $\forall x \exists y \forall z(xy=xz)$ (Prime numbers)

(n) $\forall x \exists y(x \geq 0 \Rightarrow y^2=x)$ (Real numbers)

(o) $\forall x[x<0 \Rightarrow \exists y(y^2=x)]$ (Real numbers)

(p) $\forall x[x<0 \Rightarrow \exists y(y^2=x)]$ (Positive real numbers)

II. Negate each of the statements in question **I**, putting your answers in positive form.

2.5 Determine the truth value of each statement, assuming that x, y, and z are real numbers.

(a) $\exists\, x \ni \forall\, y\, \exists\, z \ni x + y = z$.
(b) $\exists\, x \ni \forall\, y$ and $\forall\, z$, $x + y = z$.
(c) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni y - z = x$.
(d) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni xz = y$.
(e) $\exists\, x \ni \forall\, y$ and $\forall\, z$, $z > y$ implies that $z > x + y$.
(f) $\forall\, x$, $\exists\, y$ and $\exists\, z \ni z > y$ implies that $z > x + y$.

2.6 Determine the truth value of each statement, assuming that x, y and z are real numbers.

(a) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni x + y = z$.
(b) $\forall\, x\, \exists\, y \ni \forall\, z, x + y = z$.
(c) $\exists\, x \ni \forall\, y$, $\exists\, z \ni xz = y$.
(d) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni yz = x$.
(e) $\forall\, x\, \exists\, y \ni \forall\, z$, $z > y$ implies that $z > x + y$.
(f) $\forall\, x$ and $\forall\, y$, $\exists\, z \ni z > y$ implies that $z > x + y$.

ANSWERS TO PRACTICE PROBLEMS 1 and 2

(a) $\exists\, x > 0 \ni x^2 = 5$.
(b) $\forall\, M > 0\, \exists\, N > 0 \ni N < 1/M$.
(c) $\forall\, n$, if $n \geq N$, then $\forall\, x$ in A, $|f_n(x) - f(x)| \leq 3$.
(d) The words "no" and "none" are universal quantifiers in a negative sense. In general, the statement "None of them are $P(x)$" is equivalent to "All of them are not $P(x)$." Thus the statement can be written as "$\forall\, x > 0,\ f(x) \neq 5$."

(a) $\forall\, x > 0, x^2 \neq 5$.
(b) $\exists\, M > 0 \ni \forall\, N > 0, N \geq 1/M$.
(c) $\exists\, n \ni n \geq N$ and $\exists\, x$ in $A \ni |f_n(x) - f(x)| > 3$.
(d) $\exists\, x > 0 \ni f(x) = 5$.

Exercises

1. Mark each statement True or False. Justify each answer.

a To prove "$\forall\, n,\ p(n)$" is true, it takes only one example.
b To prove "$\exists\, n \ni p(n)$" is true, it takes only one example.

c The contrapositive of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

d The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
e To prove "$\forall\, n,\ p(n)$" is false, it takes only one counterexample.
f To prove "$\exists\, n \ni p(n)$" is false, it takes only one counterexample.

2. Provide a counterexample for each statement.

(a) For every real number x, if $x^2 > 4$ then $x > 2$.
(b) For every positive integer n, $n^2 + n + 41$ is prime.
(c) Every triangle is a right triangle.
(d) No integer greater than 100 is prime.
(e) Every prime is an odd number.
(f) For every positive integer n, $3n$ is divisible by 6.
(g) If x and y are unequal positive integers and xy is a perfect square, then x and y are perfect squares.
(h) Every real number has a reciprocal.
(i) For all real numbers $x > 0$, we have $x^2 \le x^3$.
(j) The reciprocal of a real number $x \ge 1$ is a real number y such that $0 < y < 1$.
(k) $3^n + 2$ is prime for all positive integers n.
(l) No rational number satisfies the equation $x^3 + (x-1)^2 = x^2 + 1$.
(m) No rational number satisfies the equation $x^4 + (1/x) - \sqrt{x+1} = 0$.

3. Suppose p and q are integers. Recall that an integer m is even iff $m = 2k$ for some integer k and m is odd iff $m = 2k + 1$ for some integer k. Prove the following. [You may use the fact that the sum and product of integers is again an integer.]

(a) If p is odd and q is odd, then $p + q$ is even.
(b) If p is odd and q is odd, then pq is odd.
(c) If p is odd and q is even, then $p + q$ is odd.
(d) If p is even and q is even, then $p + q$ is even.
(e) If p is even or q is even, then pq is even.
(f) If pq is odd, then p is odd and q is odd.
(g) If p^2 is even, then p is even. ☆
(h) If p^2 is odd, then p is odd.