Sets. Chapter 6

In supplementary exercises 5.25 and 5.26, one sees $\bigcup_{B\in\mathcal{B}} B$ and $\bigcap_{B\in\mathcal{B}} B$. These may also be written as $\displaystyle\bigcup_{B\in\mathcal{B}} B$ and $\displaystyle\bigcap_{B\in\mathcal{B}} B$ or simply as $\bigcup\mathcal{B}$ and $\bigcap\mathcal{B}$.

If the sets in $\mathcal{B}$ are indexed as, for example, B_i $i \in I$ then one may see $\displaystyle\bigcup_{i\in I} B_i$ and $\displaystyle\bigcap_{i\in I} B_i$. If $I = \mathbf{N}$ one may see $\displaystyle\bigcup_{i=1}^{\infty} B_i$ and $\displaystyle\bigcap_{i=1}^{\infty} B_i$.

Exercise 20 *In supplementary exercise 5.25 (a), (b) and (c), write* $\displaystyle\bigcup_{B\in\mathcal{B}} B$ *and* $\displaystyle\bigcap_{B\in\mathcal{B}} B$ *in the other forms defined above.*

Exercise 21 *Let S be the set of all sets that do not contain themselves as elements. ($A \in S \iff A \notin A$.) Is $S \in S$? Is $S \notin S$? Here's a popular version of this: In a town is a barber who shaves every man who does not shave himself. Who shaves the barber?*

THEOREM

Let $\{A_i \mid i \in I\}$ be an indexed family of sets, and let B be any set, all subsets of some universal set U. Then:

(1) $B \cup \displaystyle\bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \cup A_i)$

(2) $B \cap \displaystyle\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B \cap A_i)$

(3) $B \cap \displaystyle\bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \cap A_i)$

(4) $B \cup \displaystyle\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B \cup A_i)$

(5) $\left(\displaystyle\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$

(6) $\left(\displaystyle\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$

(7) *Let $\{A_i \mid i \in I\}$ be an indexed family of sets. Then for any $i_0 \in I$*

$$A_{i_0} \subseteq \bigcup_{i\in I} A_I \quad \textit{and} \quad \bigcap_{i\in I} A_I \subseteq A_{i_0}$$

5.8 Let $S = \{\varnothing, \{\varnothing\}\}$. Determine whether each of the following is True or False. Explain your answers.

(a) $\varnothing \subseteq S$ (b) $\varnothing \in S$ (c) $\{\varnothing\} \subseteq S$ (d) $\{\varnothing\} \in S$

5.9 Fill in the blanks in the following proof.

THEOREM: Let A be a subset of U. Then $A \cup (U \setminus A) = U$.

Proof: If $x \in A \cup (U \setminus A)$, then $x \in$ __________ or $x \in$ __________. Since both A and $U \setminus A$ are subsets of U, in either case we have __________. Thus __________ $\subseteq$ __________.

On the other hand, suppose that $x \in$ __________. Now either $x \in A$ or $x \notin A$. If $x \notin A$, then $x \in$ __________. In either case $x \in$ __________. Hence __________ $\subseteq$ __________. ♦

5.10 Fill in the blanks in the proof of the following theorem.

THEOREM: $A \subseteq B$ iff $A \cup B = B$.

Proof: Suppose that $A \subseteq B$. If $x \in A \cup B$, then $x \in A$ or $x \in$ __________. Since $A \subseteq B$, in either case we have $x \in B$. Thus __________ $\subseteq$ __________. On the other hand, if $x \in$ __________, then $x \in A \cup B$, so __________ $\subseteq$ __________. Hence $A \cup B = B$.

Conversely, suppose that $A \cup B = B$. If $x \in A$, then $x \in$ __________. But $A \cup B = B$, so $x \in$ __________. Thus __________ $\subseteq$ __________. ♦

5.11 Fill in the blanks in the proof of the following theorem.

THEOREM: $A \subseteq B$ iff $A \cap B = A$.

Proof: Suppose that $A \subseteq B$. If $x \in A \cap B$, then clearly $x \in A$. Thus $A \cap B \subseteq A$. On the other hand, ____.
Thus $A \subseteq A \cap B$, and we conclude that $A \cap B = A$.

Conversely, suppose that $A \cap B = A$. If $x \in A$, then ____.
Thus $A \subseteq B$. ♦

5.12 Suppose you are to prove that set A is a subset of set B. Write a reasonable beginning sentence for the proof, and indicate what you would have to show in order to finish the proof.

5.15 Which statement(s) below would enable one to conclude that $x \in A \cap B$?

(a) $x \in A$ and $x \in B$. (b) $x \in A$ or $x \in B$.
(c) $x \in A$ and $x \notin A \setminus B$. (d) If $x \in A$, then $x \in B$.

5.16 Which statement(s) below would enable one to conclude that $x \in A \setminus B$?

(a) $x \in A$ and $x \notin B \setminus A$. $= \varnothing$ (b) $x \in A \cup B$ and $x \notin B$.
(c) $x \in A \cup B$ and $x \notin A \cap B$. (d) $x \in A$ and $x \notin A \cap B$.

5.17 Which statement(s) below would enable one to conclude that $x \notin A \setminus B$?

(a) $x \notin A \cup B$. (b) $x \in B \setminus A$.
(c) $x \in A \cap B$. (d) $x \in A \cup B$ and $x \notin A$.
(e) $x \in A \cup B$ and $x \in A \cap B$.

5.19 Prove: If $U = A \cup B$ and $A \cap B = \varnothing$, then $A = U \setminus B$.

5.20 Prove: $A \cap B$ and $A \setminus B$ are disjoint and $A = (A \cap B) \cup (A \setminus B)$.

5.21 Prove or give a counterexample: $A \setminus (A \setminus B) = B \setminus (B \setminus A)$.

5.22 Prove or give a counterexample: $A \setminus (B \setminus A) = B \setminus (A \setminus B)$.

5.23 Let A and B be subsets of a universal set U. Prove the following.

(a) $A \backslash B = (U \backslash B) \backslash (U \backslash A)$

(b) $U \backslash (A \backslash B) = (U \backslash A) \cup B$

(c) $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$

5.25 Find $\bigcup_{B \in \mathscr{B}} B$ and $\bigcap_{B \in \mathscr{B}} B$ for each collection $\mathscr{B}$.

(a) $\mathscr{B} = \left\{\left[1, 1+\frac{1}{n}\right] : n \in \mathbb{N}\right\}$

(b) $\mathscr{B} = \left\{\left(1, 1+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(c) $\mathscr{B} = \{[2, x] : x \in \mathbb{R} \text{ and } x > 2\}$

(d) $\mathscr{B} = \{[0,3], (1,5), [2,4)\}$

5.26 Let $\{A_j : j \in J\}$ be an indexed family of sets and let B be a set. Prove the following generalizations of Theorem 5.13.

(a) $B \cup \left[\bigcap_{j \in J} A_j\right] = \bigcap_{j \in J} (B \cup A_j)$

(b) $B \cap \left[\bigcup_{j \in J} A_j\right] = \bigcup_{j \in J} (B \cap A_j)$

(c) $B \backslash \left[\bigcup_{j \in J} A_j\right] = \bigcap_{j \in J} (B \backslash A_j)$

(d) $B \backslash \left[\bigcap_{j \in J} A_j\right] = \bigcup_{j \in J} (B \backslash A_j)$