Math 285: Exam II

1. (10 points) Find the general solution of the homogeneous differential equation

$$y'' + 10y' + 25y = 0$$

Aux. Eqn: $m^2 + 10m + 25 = (m+5)^2 = 0$

$$y(x) = c_1 e^{-5x} + c_2 x e^{-5x}$$

2. (10 points) Find the general solution of the homogeneous differential equation

$$y'' + 6y' + 13y = 0$$

$$m^2 + 6m + 13 = 0$$

$$m = \frac{-6 \pm \sqrt{36 - 4 \cdot 1 \cdot 13}}{2} = \frac{-6 \pm \sqrt{16}\, i}{2} = -3 \pm 2i$$

$$y(x) = C_1 e^{-3x} \cos 2x + C_2 e^{-3x} \sin 2x$$

3. (10 points) Using the method of undetermined coefficients, find the general solution of the non-homogeneous differential equation (compare to the DE above)

$$y'' + 6y' + 13y = 2 + e^x$$

From (2) $y_c(x) = C_1 e^{-3x} \cos 2x + C_2 e^{-3x} \sin 2x$

Guess $y_p(x) = A + Be^x$

$$(A + Be^x)'' + 6(A + Be^x)' + 13(A + Be^x) = 2 + e^x$$

$$20Be^x + 13A = 2 + e^x$$

Thus $B = 1/20$, $A = \frac{2}{13}$

4. (10 points) Consider the non-homogeneous differential equation $y'' + y = \sin(x)$. In the context of the method of undetermined coefficients, decide whether $y_p(x) = A\cos(x) + B\sin(x)$ is a good guess for a particular solution. You do not need to determine A and B. If it is not a good guess, suggest a better guess.

$y_p(x) = A\cos(x) + B\sin(x)$ is contained in the complementary space $y_c(x) = c_1\cos(x) + c_2\sin(x)$.
A better guess is $y_p(x) = Ax\cos(x) + Bx\sin(x)$.

5. (10 points) Suppose y_1 and y_2 are solutions of a second order differential equation $ay'' + by' + cy = 0$ on the interval $I = (-\infty, \infty)$. Suppose $y_1(3)y_2'(3) - y_2(3)y_1'(3) = 0$. What can you say about the two solutions y_1 and y_2?

This is $W(y_1, y_2)$ evaluated at 3.
Since the wronskian is zero on I, the solutions $\{y_1, y_2\}$ are not fundamental.

6. (15 points) Find the general solution of the Cauchy-Euler differential equation

$$x^2y'' - 2xy' - 4y = 0$$

$$x^2(x^m)'' - 2x(x^m)' - 4x^m = 0$$

$$x^m m(m-1) - 2x^m m - 4x^m = 0$$

$$x^m(m(m-1) - 2m - 4) = 0$$

$$m^2 - 3m - 4 = 0$$

$$(m-4)(m+1) = 0$$

so $\boxed{y(x) = c_1x^4 + c_2x^{-1}}$

7. (15 points) One solution of the differential equation $xy'' + y' = 0$ is $y_1(x) = 1$. Find a second solution using the *reduction of order* formula from section 4.2. You must show your work and the formula **very clearly and very carefully** for credit.

$$y_2 = y_1 u \quad \text{where} \quad u(x) = \int \frac{e^{-\int P dx}}{(y_1)^2} dx = \int \frac{e^{-\int \frac{dx}{x}}}{(1)} dx$$

$$= \ln(x)$$

So $\boxed{y_2(x) = (1)\ln(x) = \ln(x)}$

8. (10 points) Here are two possible particular solutions

(a) $y_{p_1}(x) = x^2 \sin(3x) - 47\cos(5x)$ (b) $y_{p_2}(x) = x^2 \sin(3x) - 47\cos(5x) + e^{3x}$

of the differential equation

$$y'' - 5y' + 6y = g(x)$$

Explain why the two solutions are either both correct or both incorrect. You don't need to know $g(x)$ to answer this.

$y_{p_1}(x) - y_{p_2}(x) = e^{3x}$ which is in the complementary sol'n space.

9. (10 points) Consider the initial value problem where a, b, c are constants

$$\begin{cases} ay'' + by' + cy = g(x) \\ y(0) = 4 \\ y'(0) = 8 \end{cases}$$

If $y_c(x) = C_1 e^x + C_2 e^{2x}$ and $y_p(x) = 4x + 12$, then solve the I.V.P. Note: you do **not** need to discover the values of a, b, c and $g(x)$ to answer this question. The solution is really more straightforward than that.

$$y(x) = y_c(x) + y_p(x)$$
$$y'(x) = C_1 e^x + 2C_2 e^{2x} + 4$$

$$\begin{cases} C_1 + C_2 + 12 = 4 \\ C_1 + 2C_2 + 4 = 8 \end{cases} \implies \begin{cases} C_1 + C_2 = -8 \\ C_1 + 2C_2 = 4 \end{cases}$$

$\boxed{C_2 = 12 \text{ and } C_1 = -20}$