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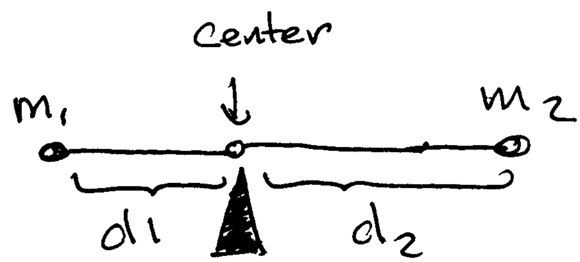
Law of the Lever

or

"Seesaw"



The "seesaw" will balance if

$$m_1 d_1 = m_2 d_2$$

If $\bar{x}$ is the location of the "center," x_1 is the location of m_1 and x_2 is the location of m_2,

then $$m_1 \underbrace{(\bar{x} - x_1)}_{d_1} = m_2 (x_2 - \bar{x})$$

Solving for $\bar{x}$,

$$(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

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$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

If the masses is distributed smoothly according to density $\delta(x)$, over an interval $[a\ b]$,

then $$\bar{x} = \frac{\int x\,\delta(x)\,dx}{\int \delta(x)\,dx}$$

density function: $\delta(x,y) = \dfrac{\text{mass}}{\text{area}}$

<u>mass of a planar plate</u>: region R

$$M = \iint_R \delta(x,y)\,dA$$

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density function $\delta(x,y,z) = \dfrac{\text{mass}}{\text{volume}}$

<u>Mass of a solid in space</u>: solid D

$$M = \iiint \delta(x,y,z)\,dV$$

$$M_x = \text{moment about } \overbrace{x\text{-axis}}^{\text{"the pivot"}}$$

$$= \iint_R y\,\delta(x,y)\,dA$$

$$M_y = \iint_R x\,\delta(x,y)\,dA$$

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ex: Find the center of mass of a triangular plate with vertices $(0,0)$, $(1,0)$ and $(0,2)$ if the density function is $\rho(x,y) = 1+3x+y$.

Sol'n:

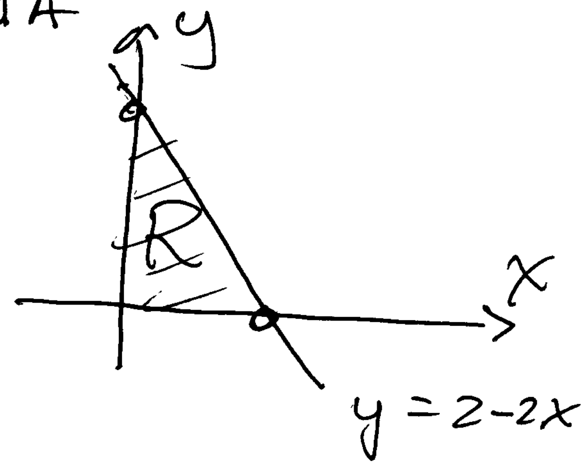
$$M = \iint_D \rho(x,y)\,dA = 8/3$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = 33/128$$

$$M_y = \iint x\,\rho(x,y)\,dA = 1$$

$$\bar{x} = M_y/M \quad \text{and} \quad \bar{y} = M_x/M$$

$$M = \iint_R \overbrace{\rho(x,y)}^{\text{units } \boxed{\frac{\text{mass}}{\text{area}}}}\, dA$$



$$M = \int_0^1 \int_0^{2-2x} (1+3x+y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_0^{2-2x} dx$$

$$= 4\int_0^1 (1 - x^2)\, dx$$

$$= 4\left[x - \frac{x^3}{3} \right]_0^1$$

$$= 8/3 \quad \boxed{\left(\frac{\text{mass}}{\text{area}}\right) \cdot \text{Area} = \text{a mass}}$$

$$M_y = \iint \overset{\text{lever}}{x} \, (\overbrace{1+3x+y}^{\text{mass density}}) \, dA$$

denotes pivot

$$= \int_0^1 \int_0^{2-2x} x(1+3x+y)\,dy\,dx$$

$$= \cdots$$

$$= 4\int_0^1 (x - x^3)\,dx$$

$$= 1$$

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Solids in xyz-space

$$M = \iiint_D \delta \, dV$$

$$M_{xy} = \iiint_D z \, \delta \, dV$$

moment about xy-plane

$$M_{xz} = \iiint_D y \, \delta \, dV$$

$$M_{yz} = \iiint_D x \, \delta \, dV$$

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$