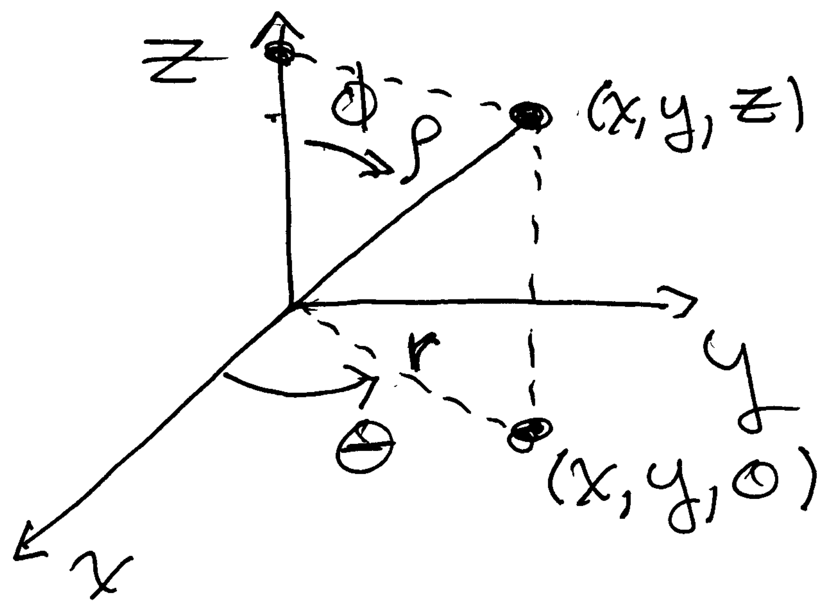
Triple Integrals

13.7 Spherical coordinates

$$(x, y, z) \longleftrightarrow (\rho, \phi, \theta)$$



$$\begin{cases} \dfrac{r}{\rho} = \sin\phi \\ \dfrac{z}{\rho} = \cos\phi \end{cases} \quad \text{or} \quad \begin{cases} r = \rho\sin\phi \\ z = \rho\cos\phi \end{cases}$$

Thus (multiply eq once by $\cos\theta$)

$$x = r\cos\theta = \rho\cos\theta\sin\phi$$
$$y = r\sin\theta = \rho\sin\theta\sin\theta$$
$$z = \rho\cos\phi.$$

Summary

$$\left[\begin{aligned} x &= \rho \cos\theta \sin\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\phi \\ \rho^2 &= x^2+y^2+z^2 \end{aligned}\right.$$

dV in spherical is

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

ex: Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$

Where B is the unit Ball centered at the origin.

Solution:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{\rho^3}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Note: In previous example, the rectangular coordinate integral would have been:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

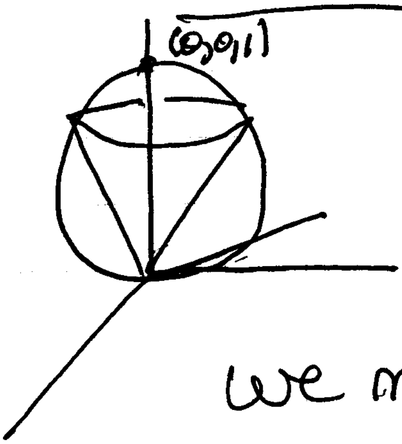
13.7

ex: Use spherical coordinates to find the volume of the solid that lies above

$$z = \sqrt{x^2+y^2}$$

and below the sphere

$$x^2+y^2+z^2 = z.$$



$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} 1\ \rho \sin^2\phi \, d\rho \, d\phi \, d\theta$$

we need a formula for ρ in terms of θ and ϕ.

The rest is obvious

$$x^2+y^2+z^2=z$$

$$\rho^2 = z$$

$$\rho^2 = \rho\cos\phi$$

$$\rho = \cos\phi \quad \text{yeah!}$$