

MATH 430/630 Exam I

- (1) (8 points) Consider the vector space $V = \mathbb{R}^3$. Decide whether the subset of V specified by the given condition is a subspace of V . If it is a subspace, you must show the closure properties hold. If it is not a subspace you must give an example that shows that at least one of the closure properties does not hold.

- (a) All vectors $[a_1 \ a_2 \ a_3]^T$ in V such that $a_1 = a_3$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_1 + b_1 \end{bmatrix} \text{ is in subset}$$

$$\alpha \begin{bmatrix} a_1 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_1 \end{bmatrix} \text{ is in the subset}$$

} Since the closure properties hold the subset is a subspace.

- (b) All vectors $[a_1 \ a_2 \ a_3]^T$ in V such that $a_1 a_3 = 0$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so the subset is not closed under addition. Not a subspace

- (2) (10 points) Exhibit bases for the range and null space of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

- (3) (8 points) Decide whether the given function from \mathbb{R}^2 to \mathbb{R} is a linear transformation. If it is a linear transformation, you must show the required properties hold in general. If it is not a linear transformation you must give an example that shows that at least one of the required properties does not hold in general.

(a) $T(x, y) = x + 1$

$$T[2(x, y)] = T(2x, 2y) = 2x + 1$$

$$\neq 2T(x, y) = 2x + 2$$

So not linear.

(b) $T(x, y) = x - y$

Linear

$$(a) \quad T[(x_1, y_1) + (x_2, y_2)] = (x_1 + x_2) - (y_1 + y_2)$$

$$= x_1 - y_1 + x_2 - y_2 = T(x_1, y_1) + T(x_2, y_2)$$

$$(b) \quad T[\alpha(x, y)] = \alpha x - \alpha y = \alpha(x - y) = \alpha T(x, y)$$

- (4) (10 points) Consider the basis $S = \{t + 2, t + 1, t^2 - 3t - 3\}$ for the vector space of polynomials of degree less than or equal to 2. Find $[at^2 + bt + c]_S$.

Let $\alpha_1 = t + 2, \alpha_2 = t + 1, \alpha_3 = t^2 - 3t - 3$

Notice $1 = \alpha_1 - \alpha_2, t = 2\alpha_2 - \alpha_1, t^2 = \alpha_3 + 3\alpha_2$

$$at^2 + bt + c = a(\alpha_3 + 3\alpha_2) + b(2\alpha_2 - \alpha_1) + c(\alpha_1 - \alpha_2)$$

$$= a\alpha_3 + (3a + 2b - c)\alpha_2 + (c - b)\alpha_1$$

$$[at^2 + bt + c]_S = a[\alpha_3]_S + (3a + 2b - c)[\alpha_2]_S + (c - b)[\alpha_1]_S$$

$$= \begin{pmatrix} c - b \\ 3a + 2b - c \\ a \end{pmatrix}$$

- (5) (6 points) Find a condition (equation) involving the components of $\vec{v} = [b_1 \ b_2 \ b_3]^T$ such that when the condition is satisfied, \vec{v} will be in the span of

$$T = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 10 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 2 & 3 & 7 & b_2 \\ 3 & 4 & 10 & b_3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & 5 & 5 & b_2 - 2b_1 \\ 0 & 7 & 7 & b_3 - 3b_1 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & 35 & 35 & 7b_2 - 14b_1 \\ 0 & 35 & 35 & 5b_3 - 15b_1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & 35 & 35 & 7b_2 - 14b_1 \\ 0 & 0 & 0 & 5b_3 - 7b_2 - b_1 \end{array} \right]$$

condition $5b_3 - 7b_2 - b_1 = 0$ //

- (6) (8 points) Consider the basis $S = \{1, t + 2, t^2 + t\}$ for the vector space of polynomials of degree less than or equal to 2. Let T be given by

$$T(at^2 + bt + c) = at^2 + (2b - a)t + (3c - 2b + 2a).$$

Find the coordinate matrix $[T]_S$ for T .

$$[T]_S = \left([T(1)]_S \mid [T(t+2)]_S \mid [T(t^2+t)]_S \right) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T(1)]_S = [3 \cdot 1]_S = 3[1]_S$$

$$[T(t+2)]_S = [2t + 4]_S = [2 \cdot (t+2)]_S = 2[t+2]_S$$

$$[T(t^2+t)]_S = [t^2+t]_S \quad \text{column 3}$$