A plane in space is determined by knowing a point on the plane and knowing its "tilt."

\[ \vec{n} = \text{normal} = \text{"tilt"} \]

\[ \vec{n} \text{ has the property that} \]

1) it is perpendicular to all lines contained in the plane

2) it is perpendicular to all vectors parallel to the plane.

\[ \vec{n} \cdot \vec{P_0P} = 0 \]

\[ \langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0 \]

\[ d = ax + by + cz + (\overline{-ax_0-by_0-cz_0}) = 0 \]
Its direction. Thus a plane in space is determined by a point \( P(x_0, y_0, z_0) \) in the plane and a vector \( \mathbf{n} \) that is orthogonal to the plane. This orthogonal vector \( \mathbf{n} \) is called a normal vector. Let \( P(x, y, z) \) be an arbitrary point in the plane, and let \( \mathbf{r}_0 \) and \( \mathbf{r} \) be the position vectors of \( P_0 \) and \( P \). Then the vector \( \mathbf{r} = \mathbf{r}_0 \) is represented by \( \overrightarrow{P_0P} \). (See Figure 6.) The normal vector \( \mathbf{n} \) is orthogonal to every vector in the given plane. In particular, \( \mathbf{n} \) is orthogonal to \( \mathbf{r} - \mathbf{r}_0 \) and so we have

\[
\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0
\]

which can be rewritten as

\[
\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0
\]

Either Equation 5 or Equation 6 is called a vector equation of the plane.

To obtain a scalar equation for the plane, we write \( \mathbf{n} = (a, b, c) \), \( \mathbf{r} = (x, y, z) \), and \( \mathbf{r}_0 = (x_0, y_0, z_0) \). Then the vector equation 5 becomes

\[
(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0
\]

or

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]

Equation 7 is the scalar equation of the plane through \( P_0(x_0, y_0, z_0) \) with normal vector \( \mathbf{n} = (a, b, c) \).

**Example 4** Find an equation of the plane through the point \((2, 4, -1)\) with normal vector \( \mathbf{n} = (2, 3, 4) \). Find the intercepts and sketch the plane.

**Solution** Putting \( a = 2 \), \( b = 3 \), \( c = 4 \), \( x_0 = 2 \), \( y_0 = 4 \), and \( z_0 = -1 \) in Equation 7, we see that an equation of the plane is

\[
2(x - 2) + 3(y - 4) + 4(z + 1) = 0
\]

or

\[
2x + 3y + 4z = 12
\]

To find the \( x \)-intercept we set \( y = z = 0 \) in this equation and obtain \( x = 6 \). Similarly, the \( y \)-intercept is 4 and the \( z \)-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

\[
ax + by + cz + d = 0
\]

where \( d = -(ax_0 + by_0 + cz_0) \). Equation 8 is called a linear equation in \( x, y, \) and \( z \). Conversely, it can be shown that if \( a, b, \) and \( c \) are not all 0, then the linear equation 8 represents a plane with normal vector \((a, b, c)\). (See Exercise 81.)
Consider the plane

$$5x + 2y - z = 2z.$$ What is a normal vector?
Ex. Given the points
\[ P = (0, 2, 0), \quad Q = (1, 4, 4), \quad R = (0, 2, 4) \]
on a plane, find a normal and then an equation of the plane.
Find the plane determined by the intersecting lines

\[ L_1 \begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = 3 + 2t \end{cases} \quad L_2 \begin{cases} x = 1 + 2t \\ y = 2 \\ z = 3 - t \end{cases} \]

A normal is \( \vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 0 & -1 \end{vmatrix} \)

\[ \vec{n} = (-2, 5, -4) \]

\((x_0, y_0, z_0) = (1, 2, 3)\) is on \(L_1\), so on the plane.

\[-2(x-1) + 5(y-2) - 4(z-3) = 0\]
Exercise: Find the angle between the planes $5x + y - z = 10$ and $x - 2y + 3z = -1$.

$P_1: 5x + y - z = 10$ has normal $\vec{n}_1 = \langle 5, 1, -1 \rangle = 5\hat{i} + \hat{j} - \hat{k}$

$P_2: x - 2y + 3z = -1$ has normal $\vec{n}_2 = \langle 1, -2, 3 \rangle$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = 0$$

So $\theta = \frac{\pi}{2}$ radians
Find the vector equation of the line \( L \) of intersection of the planes \( x+y+z=1 \) and \( x-2y+3z=1 \).

If \( \vec{v} \) is parallel to the line \( L \), then \( \vec{v} \perp \vec{n}_1 = <1, 1, 1> \) and \( \vec{v} \perp \vec{n}_2 = <1, -2, 3> \).

So we can choose \( \vec{v} = \vec{n}_1 \times \vec{n}_2 = <6, -2, -3> \).

A vector equation of the line is

\[
\vec{r}(t) = <x_0, y_0, z_0> + t <5, -2, -3>
\]

Where \( (x_0, y_0, z_0) \) solves\[
\begin{cases}
x + y + z = 1 \\
x - 2y + 3z = 1
\end{cases}
\]

Note: \( (x_0, y_0, z_0) = (-4, 2, 3) \) works

or \( (x_0, y_0, z_0) = (0, 0, 0) \) works

since both points are in the intersection of the two planes and so are both on \( L \).
Distance from a point \( P_i \) to a plane \( P_i(x_i, y_i, z_i) \)

Plane:
\[ ax + by + cz + d = 0 \]

\[
\vec{b} = \vec{P_0P_i} = \langle x_i - x_0, y_i - y_0, z_i - z_0 \rangle
\]

\[
D = \left| \operatorname{comp}_{\vec{n}} \vec{b} \right| = \frac{\left| \vec{n} \cdot \vec{b} \right|}{\left| \vec{n} \right|}
\]

\[
D = \left| \langle a, b, c \rangle \cdot \langle x_i - x_0, y_i - y_0, z_i - z_0 \rangle \right| \quad \frac{1}{\left| \langle a, b, c \rangle \right|}
\]

\[
D = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left| ax_i + by_i + cz_i + d \right|
\]
Example Find the distance from the point $P_1(0,1,1)$ to the plane $4y + 3z - 12 = 0$.

\[ \vec{n} = \langle a, b, c \rangle = \langle 0, 4, 3 \rangle \]

\[ D = \frac{|ax_1 + by_1 + cz_1|}{\sqrt{a^2 + b^2 + c^2}} \]

\[ = \frac{|(0)(0) + (4)(1) + (3)(1) - 12|}{\sqrt{0^2 + 4^2 + 3^2}} \]

\[ = \frac{1 - \frac{15}{5}}{5} = 1 \]
$e^x$