Definition. We say that a function \( f(x,y) \) approaches the limit \( L \) as \( (x,y) \) approaches \( (x_0,y_0) \) and write

\[
\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L
\]

if, for every number \( \varepsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \( (x,y) \) in the domain of \( f \),

\[
|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.
\]
THEOREM 1 — Properties of Limits of Functions of Two Variables

The following rules hold if $L$, $M$, and $k$ are real numbers and

$$
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} g(x, y) = M.
$$

1. Sum Rule: $$\lim_{(x, y) \to (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

2. Difference Rule: $$\lim_{(x, y) \to (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

3. Product Rule: $$\lim_{(x, y) \to (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

4. Constant Multiple Rule: $$\lim_{(x, y) \to (x_0, y_0)} (kf(x, y)) = kL \quad \text{(any number } k)$$

5. Quotient Rule: $$\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad M \neq 0$$

6. Power Rule: If $r$ and $s$ are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x, y) \to (x_0, y_0)} (f(x, y))^r = L^r$$

Provided $L^r$ is a real number. (If $s$ is even, we assume that $L > 0$.)
Ex:

(1) Find the limit of
\[
\lim_{(x,y) \to (2,3)} \sqrt{x^2 + y^2 - 1}
\]

(2) \[
\lim_{(x,y) \to (2,1)} \frac{x^2 - xy - 2y^2}{x - 2y}
\]
Continuity

As with functions of a single variable, continuity is defined in terms of limits.

**DEFINITION** A function \( f(x, y) \) is continuous at the point \((x_0, y_0)\) if

1. \( f \) is defined at \((x_0, y_0)\).
2. \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) \) exists,
3. \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0) \).

A function is continuous if it is continuous at every point of its domain.