Theorem (Fundamental Theorem for Line Integrals)

Let \( C \) be a smooth curve given by \( \mathbf{r}(t) \) for \( a \leq t \leq b \).

Let \( f \) be a differentiable function whose gradient \( \nabla f \) is continuous on \( C \). Then

\[
\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\]
Note: Since conservative vector fields are gradient fields, if $\vec{F}$ is conservative then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{\nabla} f \cdot d\vec{r} = f(B) - f(A)$$

While

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C} \vec{\nabla} f \cdot d\vec{r} = f(B) - f(A)$$

So

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

For a conservative vector field $\vec{F}$,

$\int_C \vec{F} \cdot d\vec{r}$ only depends on the initial and terminal points of the curve $C$. 
If generally, if \( \int_{C} F \cdot dr \) only depends upon the initial and terminal points of the curve \( C \) (for all curves in some region \( D \) then we say that the line integral is \underline{independent of path} in \( D \).

Note: this is really a feature of \( F \).

Def: A closed curve (path) is a curve whose initial and terminal points are the same.

\[ A = B \]
Example: Let \( \vec{F} = y\hat{i} + x\hat{j} \).

Observe that \( \vec{F} = \nabla f \) where 
\[ f(x, y) = xy. \]

Evaluate \( \int_{C_1} \vec{F} \cdot d\vec{r} \) and \( \int_{C_2} \vec{F} \cdot d\vec{r} \).
Theorem $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D$ if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path in $D$. 
Theorem: Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region $D$.

If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$, then $\mathbf{F}$ is a conservative vector field. That is, there is a scalar function $f$ such that

$$\mathbf{F} = \nabla f.$$
Theorem: If $\vec{F} = P\hat{i} + Q\hat{j}$ is a conservative vector field, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Theorem: Let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field on an open simply-connected region $D$. Suppose $P_x, P_y, Q_x, Q_y$ are continuous.

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout $D$, then $\vec{F}$ is conservative.
Ex. Determine whether the vector field is conservative.

a) \( \vec{F}_1 = \langle 2x + y, x + 2y \rangle \)

b) \( \vec{F}_2 = \langle -y, x \rangle \)
ex Let \( \mathbf{F} = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j} \)

a) Find a function \( f \) such that \( \mathbf{F} = \nabla f \)

Solution

\[ f_x = 3 + 2xy \quad (1) \]
\[ f_y = x^2 - 3y^2 \quad (2) \]

\[
\begin{align*}
  f(x, y) &= \int f_x \, dx = 3x + x^2y + g(y) \\
  f_y(x, y) &= x^2 - 3y^2 = \frac{\partial}{\partial y} (3x + x^2y + g(y)) \\
  x^2 - 3y^2 &= x^2 + g'(y) \\
  \text{Thus} \quad g'(y) &= -3y^2 \\
  g(y) &= -y^3 + K \\
  \text{Then} \quad f(x, y) &= 3x + x^2y - y^3 + K
\end{align*}
\]
ex. Again, let \( \vec{F} = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j} \)

b) Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) where 

\( C \) is the curve given by 

\[ \vec{r}(t) = e^t \sin t \vec{i} + e^t \cos t \vec{j}, \quad 0 \leq t \leq \pi. \]
ex: Let \( \mathbf{F} = \langle yz, xz, \cos z + xy \rangle \)

Try to find a scalar function \( f(x, y, z) \) such that \( \mathbf{F} = \nabla f \).

1. \( f_x = yz \)
2. \( f_y = xz \)
3. \( f_z = \cos(z) + xy \)

\[ f(x, y, z) = \int f_x \, dx = xyz + g(y, z) \]

\[ f_y = \frac{\partial}{\partial y} (xyz + g(y, z)) \]

\[ xz = xz + g_y(y, z) \] so \( g_y(y, z) = 0 \)

Thus \( g(y, z) = h(z) \)

\[ f(x, y, z) = xyz + h(z) \]

\[ \cos(z) + xy = f_z = \frac{\partial}{\partial z} (xyz + h(z)) \]

\[ \cos(z) + xy = xy + h'(z) \]

\( h'(z) = \cos(z) \) so \( h(z) = \sin(z) + k \)

\[ f(x, y, z) = xyz + \sin(z) + k \]