Section 5.2 Discrete

we again consider the discrete dynamical system $\Delta X = F(X(n))$ where $\Delta X \overset{\text{def}}{=} X(n+1) - X(n)$.

A sequence $\{X(n)\}_{n=1}^{\infty}$ is a solution precisely when

$X(n+1) - X(n) = F(X(n))$ for $n = 1, 2, \ldots$

or $X(n+1) = X(n) + F(X(n))$ for $n = 1, 2, \ldots$

we define the iteration function $G$ by

$G(X(n)) = X(n) + F(X(n))$.

Then $\{X(n)\}_{n=1}^{\infty}$ is a solution of the discrete system precisely when

$X(2) = G(X(1)), \ X(3) = G(X(2)), \ \ldots, \ X(n+1) = G(X(n))$. 

Let \( x_0 \) be a steady state solution.

Consider \( G(x) = (g_1(x), g_2(x)) \) and \( x \in \mathbb{R}^2 \).

Define \( A = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_0) & \frac{\partial g_1}{\partial x_2}(x_0) \\ \frac{\partial g_2}{\partial x_1}(x_0) & \frac{\partial g_2}{\partial x_2}(x_0) \end{pmatrix} \).

If all the eigenvalues of \( A \) have modulus less than 1, then \( x_0 \) is stable.

If \( G(x) \) is itself linear then we can say \( x_0 \) is stable if and only if all of the eigenvalues have modulus less than 1.

Note: If \( \lambda = a + bi \), the modulus is \( |a + bi| = \sqrt{a^2 + b^2} \).
Define $\Delta x = -\lambda x$ so $F(x) = -\lambda x$ and $G(x) = x + F(x) = x - \lambda x$ where $x \in \mathbb{R}^2$. Clearly $x_0 = (0,0)$ is a steady state solution.

$A = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$ and the eigenvalues of $A$ are $1-\lambda$. $x_0 = (0,0)$ is a stable steady state solution if $-1 < 1-\lambda < 1$ or $-1 < \lambda - 1 < 1$ or $0 < \lambda < 2$. 
Define \( \Delta x = \lambda (x_2, x_1) \) so that
\[
G(x) = (g_1(x), g_2(x)) = (x_1, x_2) + \lambda (x_2, x_1)
\]
\[
= (x_1 + \lambda x_2, x_2 + \lambda x_1)
\]
Clearly \( x_0 = (0, 0) \) is a steady state solution.

Then \( A = \begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2}
\end{pmatrix} = \begin{pmatrix}
1 & \lambda \\
\lambda & 1
\end{pmatrix}
\]

The characteristic polynomial of \( A \) is
\[p(x) = (x-1)(x-1) - \lambda^2 = x^2 - 2x + 1 - \lambda^2\]
so the eigenvalues (which are the zeros of \( p(x) \)) are \(-\lambda + 1\) and \(\lambda + 1\).

There is no \( \lambda \) for which both eigenvalues are less than 1. Thus this linear system is unstable.