6.3 Euler method: Single first order ODE

Context: We have an initial value problem
\[
\begin{aligned}
    y' &= f(t, y(t)) \\
    y(t_0) &= y_0.
\end{aligned}
\]  

(1)

Recall the Taylor expansion for a sufficiently nice function \( y \) of \( t \),
\[
y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n + 1)!} y^{(n+1)}(\xi_i)
\]

(2)

where \( h = t_{i+1} - t_i \) and \( \xi \in (t_i, t_{i+1}) \). By letting \( n = 1 \) we get
\[
y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!} y^{(2)}(\xi_i).
\]

which upon truncation of the term \( E_1 := \frac{h^2}{2!} y^{(2)}(\xi_i) \) yields the Euler method
\[
y_{i+1} = y_i + hf(t_i, y_i).
\]

(3)
Consider $n$ iterations of the Euler method

- starting at $t_0 = a,$
- moving in steps of length $h = (b - a)/n,$
- and stopping at $t_n = b.$

We then have

$$y_1 = y_0 + hf(t_0, y_0)$$
$$y_2 = y_1 + hf(t_1, y_1)$$
$$y_3 = y_2 + hf(t_2, y_2)$$
$$\vdots$$
$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}).$$

Since $E_1 := \frac{h^2}{2^1} y^{(2)}(\xi_i) \text{ is a multiple of } h^2,$ we say the local error of Euler’s method is on the order of $h^2.$ We also write $O(h^2).$

If we apply Euler’s method over an interval $[a, b]$ with $n$ iterations so that $h = (b - a)/n,$ then the cumulative error is roughly $1/h$ times the local error. So it seems reasonable that the global error (the error over the entire interval $[a, b]$) should be $O(h).$
1 Euler method: First order systems

Context: We have an initial value problem

\[
\begin{align*}
    x'(t) &= f_1(t, x(t), y(t)) \\
    y'(t) &= f_2(t, x(t), y(t)) \\
    x(t_0) &= x_0 \\
    y(t_0) &= y_0.
\end{align*}
\]

In this section we discuss the Euler method for numerically solving systems such as the one above. Recall the Taylor expansion for sufficiently nice functions \(x\) and \(y\) of \(t\),

\[
\begin{align*}
    x(t_{i+1}) &= x(t_i) + hx'(t_i) + \frac{h^2}{2!}x^{(2)}(\eta_i) \\
    y(t_{i+1}) &= y(t_i) + hy'(t_i) + \frac{h^2}{2!}y^{(2)}(\xi_i)
\end{align*}
\]

which upon truncation of the error terms \(\frac{h^2}{2!}x^{(2)}(\eta_i)\) and \(\frac{h^2}{2!}y^{(2)}(\xi_i)\) yields the Euler method

\[
\begin{align*}
    x_{i+1} &= x_i + hx'_i \\
    y_{i+1} &= y_i + hy'_i.
\end{align*}
\]

Recalling the context, \(x'(t) = f_1(t, x(t), y(t))\) and \(y'(t) = f_2(t, x(t), y(t))\) we obtain

\[
\begin{align*}
    x_{i+1} &= x_i + hf_1(t_i, x_i, y_i) \\
    y_{i+1} &= y_i + hf_2(t_i, x_i, y_i)
\end{align*}
\]

which is the Euler method for a 2 \(\times\) 2 system.